8. König's Theorem

The total kinetic energy of a system of particles is

\[ E_{\text{K}} = \frac{1}{2} \sum m_i v_i^2 \]

a) Derive an expression of this form

\[ E_{\text{K}} = \frac{1}{2} m_{\text{tot}} v_0^2 + \left[ \ldots \right] \]

First, let's define the system of particles with a center of mass:

\[ v_{i/0} = v_{0/0} + v_{i/0} \]

Subbing this into the given statement,

\[ E_{\text{K}} = \frac{1}{2} \sum m_i \left( v_{0/0} + v_{i/0} \right)^2 \]

\[ = \frac{1}{2} \sum m_i \left( v_{0/0}^2 + v_{0/0} v_{i/0} + v_{i/0}^2 \right) \]

\[ = \frac{1}{2} m_{\text{tot}} v_0^2 + \frac{1}{2} v_0 \sum m_i v_{i/0} + \frac{1}{2} \sum m_i v_{i/0}^2 \]
If \( M_{\text{tot}} \vec{v}_G = \Sigma m_i \vec{v}_i \)

\[ \Rightarrow \Sigma m_i \vec{v}_i/G = M_{\text{tot}} \vec{v}_G/G = 0 \]

because the relative velocity of G with respect to G must be zero.

\[ \Rightarrow \]

SOLUTION:

\[ E_k = \frac{1}{2} M_{\text{tot}} v_G^2 + \frac{1}{2} \Sigma m_i v_i^2/G \]

\[ \checkmark \]
b) Is it always true that
\[
(E^2_{\text{ext}}) \cdot \vec{V}_G = \frac{d}{dt} \left( \frac{1}{2} M_{\text{tot}} V^2_G \right)
\]

We know that \( E^2_{\text{ext}} = \Sigma m_i \vec{a}_i \cdot \vec{v}_0 \)

\[= M_{\text{tot}} \cdot \vec{a}_G \cdot \vec{V}_G \]

\[= M_{\text{tot}} \cdot \vec{V}_G \cdot \vec{V}_G = \frac{d}{dt} \left( \frac{1}{2} M_{\text{tot}} \vec{V}_G \cdot \vec{V}_G \right)\]

Differentiating w/ chain rule,

\[= M_{\text{tot}} \cdot \vec{V}_G \cdot \vec{V}_G = \frac{1}{c} \left( M_{\text{tot}} (\vec{V}_G \cdot \vec{V}_G + \vec{V}_G \cdot \vec{V}_G) \right)\]

**SOLUTION**

\[M_{\text{tot}} \cdot \vec{V}_G \cdot \vec{V}_G = M_{\text{tot}} \cdot \vec{V}_G \cdot \vec{V}_G\]

So yes, it is always true.
c) Is it always true that the power due to internal forces is equal to the rate of change of the quantity you filled in for (part a).
so, let's create a system where there are internal forces, but some of the particles
are constrained, and see if its power is equal to $E \cdot i / 6 \cdot V / 10$:

I got the idea to do this from office hours.

The spring is in tension, then released.

FBD:

For $M_1$, the system is constrained by the wire: $T = F_{21}$, and $M_1$ does not move.

We can write out the power due to internal forces as

$$P_{\text{int}} = \bar{F}_{21} \cdot \bar{V}_{1/10} + F_{12} \cdot \bar{V}_{2/10}$$

We knew $\bar{V}_{1/10} = 0$, so

$$P_{\text{int}} = -F_{12} \cdot \bar{V}_{2/10}$$

Now, looking at our claim, we ask:

$$P_{\text{int}} = E \cdot \bar{a} / 16 \cdot \bar{V}_{1/10}$$
expanding RHS:

$$= m_1 \ddot{a}_{1/6} \cdot \dot{V}_{1/0} + m_2 \ddot{a}_{2/6} \cdot \dot{V}_{2/0}$$

Again, $\dot{V}_{1/0} = 0$, so we have

$$\vec{F}_{12} \cdot \dot{V}_{2/0} \ ? = m_2 \ddot{a}_{2/6} \cdot \dot{V}_{2/0}$$

we know $\vec{F}_{12} = m_2 \ddot{a}_{2/0}$

$$\Rightarrow m_2 \ddot{a}_{2/0} \cdot \dot{V}_{2/0} = m_2 \ddot{a}_{2/6} \cdot \dot{V}_{2/0}$$

If $\ddot{a}_{2/0} = \ddot{a}_{2/6} + \dot{a}_{6/0}$

$$\Rightarrow \ddot{a}_{2/6} \cdot \dot{V}_{2/0} + \dot{a}_{6/0} \cdot \dot{V}_{2/0} = m_2 \ddot{a}_{2/6} \cdot \dot{V}_{2/0}$$

**SOLUTION:**

This is not true, and only could be true if the second mass were not moving (which it is) or the center of gravity is not accelerating.