15) What means "rate of change of angular momentum" for a SYSTEM of particles?

For which of these definitions of $\vec{H}/c$ is the following equation of motion true: $\vec{M}_c = \vec{H}/c$?

a) $\vec{H}/c = \sum \vec{r}/c' \times \vec{v}/c' m_i$

where $c'$ is a point fixed in $\mathcal{Q}$ that instantaneously coincides with $c$

First, we know that $\vec{v}_{i/0} = \vec{v}_{c'/0} + \vec{v}_{i/c'}$

$\Rightarrow \star \vec{v}_{i/c'} = \vec{v}_{i/0} - \vec{v}_{c'/0} \star$

Subbing this into our expression for $\vec{H}/c$:

$\vec{H}/c = \sum \vec{r}/c' \times (\vec{v}_{i/0} - \vec{v}_{c'/0}) m_i$

Since $c'$ is a fixed point in $\mathcal{Q}$, $\vec{v}_{c'/0} = \vec{0}$
15) a) (continued)

\[ \vec{\mathbf{H}}_{c} = \sum \vec{r}_{i}^{c/c'} \times (\vec{v}_{i/0} - \vec{v}) \, m_{i} \]

\[ \vec{H}_{c} = \sum \vec{r}_{i}^{c/c'} \times \vec{v}_{i/0} \, m_{i} \]

\[ \Rightarrow \vec{\dot{H}}_{c} = \sum \left[ (\vec{r}_{i}^{c/c'} \times \vec{v}_{i/0} \, m_{i}) + (\vec{r}_{i}^{c/c'} \times \vec{\dot{v}}_{i/0} \, m_{i}) \right] \]

\[ \vec{\dot{H}}_{c} = \sum \left[ (\vec{v}_{i/c'} \times \vec{v}_{i/0} \, m_{i}) + (\vec{r}_{i}^{c/c'} \times \vec{\ddot{v}}_{i/0} \, m_{i}) \right] \]

Since \( C' \) is fixed in \( \mathcal{F} \) and the origin \( O \) is also fixed, the velocity of particle \( i \) with respect to \( C' \) and \( O \) will be the same

\[ \Rightarrow \vec{v}_{i/c'} = \vec{v}_{i/0} \]

\[ \Rightarrow \vec{\dot{H}}_{c} = \sum \left[ (\vec{v}_{i/0} \times \vec{v}_{i/0} \, m_{i}) + (\vec{r}_{i}^{c/c'} \times \vec{\ddot{v}}_{i/0} \, m_{i}) \right] \]

Since a vector crossed with itself is 0

\[ \Rightarrow \vec{\dot{H}}_{c} = \sum \vec{r}_{i}^{c/c'} \times \vec{\ddot{v}}_{i/0} \, m_{i} \]
15) a) (continued)

Finally, since $C'$ instantaneously coincides with $C$ we have

$$\vec{\mathbf{x}}_i / c' = \vec{\mathbf{v}}_i / c$$

$$\Rightarrow \quad \vec{\mathbf{H}} / c = \sum \vec{\mathbf{v}}_i / c \times \vec{\mathbf{a}}_i / \mu_0 \, m_i$$

It has been shown that this definition works in general
15) b) 

\[ \hat{\mathbf{H}}_c = \sum \hat{r}_{i/c} \times \hat{v}_{i/0} m_i \]

\[ \Rightarrow \hat{\mathbf{H}}_c = \sum \left[ (\hat{r}_{i/c} \times \hat{v}_{i/0} m_i) + (\hat{\mathbf{r}}_{i/c} \times \hat{\mathbf{a}}_{i/0} m_i) \right] \]

This expression collapses to \( \hat{\mathbf{H}}_c = \sum \hat{r}_{i/c} \times \hat{\mathbf{a}}_{i/0} m_i \) for some special cases:

1) If C is stationary

If C is stationary, then we know that \( \hat{v}_{i/c} = \hat{v}_{i/0} \). Since the origin O is also stationary,

\[ \Rightarrow \hat{\mathbf{H}}_c = \sum \left[ \hat{v}_{i/0} \times \hat{v}_{i/0} m_i \right] + (\hat{\mathbf{r}}_{i/c} \times \hat{\mathbf{a}}_{i/0} m_i) \]

\[ \hat{\mathbf{H}}_c = \sum \hat{r}_{i/c} \times \hat{\mathbf{a}}_{i/0} m_i \]
15) b) (continued)

2) If all particles $i$ are moving away from point $C$ in the same or opposite direction that they are moving away from point $O$

$$\Rightarrow \vec{v}_{i/c} \times \vec{v}_{i/0} = 0$$

$$\Rightarrow \text{The point } C \text{ is the origin, } O$$

$$\dot{\mu}_{1/c} = \sum \left[ (\vec{v}_{i/c} \times \vec{v}_{i/0} \text{ m}) + (\vec{v}_{i/c} \times \vec{a}_{i/0} \text{ m/s}) \right]$$

$$\dot{\mu}_{1/c} = \sum \left[ (\vec{v}_{i/0} \times \vec{v}_{i/0} \text{ m/s}) + (\vec{v}_{i/c} \times \vec{a}_{i/0} \text{ m/s}) \right]$$

$$\dot{\mu}_{1/c} = \sum \vec{v}_{i/c} \times \vec{a}_{i/0} \text{ m/s} \checkmark$$
3) All of the systems particles \( i \) are "stuck" to the point \( C \)

\[ \vec{v}_{i/c} = 0 \]

\[ \vec{H}_{i/c} = \sum (\vec{r}_{i/c} \times \vec{v}_{i/0 \, m_i}) + (\vec{r}_{i/c} \times \vec{a}_{i/0 \, m_i}) \]

\[ \vec{H}_{i/c} = \sum \vec{r}_{i/c} \times \vec{a}_{i/0 \, m_i} \quad \checkmark \]

\[ \Rightarrow \]

This definition works for some special cases concerning the motions of the particles \( i \) and point \( C \) that I have outlined previously.
15) c)

\[ \dot{\mathbf{H}}/c = \sum \dot{\mathbf{r}}_{i}/c \times \dot{\mathbf{v}}_{i}/c \; m_{i} \]

\[ \Rightarrow \dot{\mathbf{H}}/c = \sum \left[ (\dot{\mathbf{r}}_{i}/c \times \dot{\mathbf{v}}_{i}/c \; m_{i}) + (\dot{\mathbf{r}}_{i}/c \times \dot{\mathbf{v}}_{i}/c \; m_{i}) \right] \]

\[ \dot{\mathbf{H}}/c = \sum \left[ (\dot{\mathbf{v}}_{i}/c \times \dot{\mathbf{v}}_{i}/c \; m_{i}) + (\dot{\mathbf{r}}_{i}/c \times \ddot{\mathbf{r}}_{i}/c \; m_{i}) \right] \]

The expression collapses to \( \dot{\mathbf{H}}/c = \sum \dot{\mathbf{r}}_{i}/c \times \ddot{\mathbf{r}}_{i}/c \; m_{i} \) for the following cases:

1) \text{C is stationary} \quad \text{can just refer to (a)}

If \( C \) is fixed, the the velocity of particles \( i \) with respect to \( C \) will be the same as the velocity of particles with respect to the origin \( O \) (since \( O \) is also fixed)

\[ \Rightarrow \dot{\mathbf{v}}_{i}/c = \dot{\mathbf{v}}_{i}/o \]
15) c) (continued)

Taking a derivative yields that

* $\dot{\vec{a}} / c = \vec{a} / 10 *$

Thus, we have

$\dot{\vec{H}} / c = \sum \dot{\vec{r}} / c \times \dot{\vec{a}} / c \vec{m}_i$

$\dot{\vec{H}} / c = \sum \ddot{\vec{r}} / c \times \vec{a} / 10 \vec{m}_i \checkmark$

2) C is the origin, O

\[ \Rightarrow \dot{\vec{a}} / c = \vec{a} / 10 \text{ and we get} \]

$\dot{\vec{H}} / c = \sum \ddot{\vec{r}} / c \times \dot{\vec{a}} / c \vec{m}_i$

$\dot{\vec{H}} / c = \sum \ddot{\vec{r}} / c \times \vec{a} / 10 \vec{m}_i \checkmark$

AND fixed! (not just instantaneously)

So this is a special case of (21) above.
(Not needed)
3) \( C \) is the center of mass (CoM)

We know that \[ \sum \mathbf{F}^{\text{ext}} = \sum m_i \mathbf{a}_{G/0} \]

Thus, if the sum of the external forces on this system is zero (a valid assumption) we get that

\[ \mathbf{a}_{G/0} = \mathbf{0} \]

since \( \sum m_i \) will not feasibly be equal to \( 0 \)

Also, we have that

\[ \mathbf{a}_{i/0} = \mathbf{a}_{G/0} + \mathbf{a}_{i/G} \]

Differentiating twice with respect to time we get

\[ \ddot{\mathbf{a}}_{i/0} = \ddot{\mathbf{a}}_{G/0} + \ddot{\mathbf{a}}_{i/16} \]

\[ \mathbf{a}_{i/10} = \mathbf{a}_{i/16} \]
15) c) (continued)

Thus, we had

\[ \hat{p}/c = \sum \hat{p}/ic \times \hat{a}/ic m_i \]

\[ \hat{p}/c = \sum \hat{p}/ic \times \hat{a}/i/0 m_i \quad \text{(if } C = G) \]

Using the result that \( \hat{a}/i/0 = \hat{a}/i/0 G \) gives

\[ \hat{p}/c = \sum \hat{p}/ic \times \hat{a}/i/0 m_i \quad \checkmark \]

This definition works for some special cases concerning the motions of particles \( i \) and point \( C \) which have been previously outlined.

Also if \( \hat{a}_c \) is parallel to \( \hat{p}/c \).

\[ \text{[then] } \hat{p}/c \times \hat{a}_c = \hat{0} \]

This is one of the ways people get the right answers sometimes.