A Walking Model with No Energy Cost

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We have numerically found periodic collisionless motions of a walking model consisting of linked rigid objects. Unlike previous designs, this model can walk on level ground at non-infinitesimal speed with zero energy input. The model avoids collisional losses by using an internal mode of oscillation: swaying of the upper body coupled to the legs by springs. Appropriate synchronized internal oscillations set the foot-strike collision to zero velocity. The concept might be of use for energy-efficient robots and may also help to explain aspects of human and animal locomotion efficiency.

I. INTRODUCTION

When an object, robot, or animal traverses level ground at a constant speed, the essential forces from gravity and support are orthogonal to the motion. Thus, the essential energetic cost is zero. For example, zero-cost locomotion is achieved by sliding on a frictionless surface or by rolling without slip on a frictional surface.

Can legged transport over level ground be similarly energy-cost free? Nature and engineers have designed low friction hinges. Air-friction losses are small for walking. If one neglects these minor friction losses, is a zero-energy-cost walking mechanism possible?

Consider walking mechanisms made of frictionlessly-linked rigid objects (links) subject to gravity, supported by a frictional level surface, and with possible actuators and springs at the hinges (joints). If we limit attention to periodic motions, then the positive work of the actuators (muscles) is balanced by negative actuator work (e.g., *eccentric* muscle contractions), frictional sliding, and collisions between the feet and the ground. Our central concern is the collisional loss.

At a collision, energy is lost by a combination of mechanisms (heat at the collision point, dissipation in muscles and soft tissues, acoustic radiation, etc.). If the rigid-object model is accurate for all parts before and after the collision then linear and angular momentum balance determine the energy lost, regardless of the mechanisms of dissipation [1]. In particular, energy is necessarily lost when any non-massless objects stick together with an abrupt change in velocity. Collision loss is central in passive dynamic walkers [2, 3] and important in some powered robots [4]. Collisions seem to be important role in human and animal locomotion [5, 6]. However, there is no collisional loss if the new contacts are made with zero relative velocity between the approaching objects.

It is possible to specify kinematic paths for the links of a walking device so that no collisions would occur. Using a controller and actuators one might achieve that collisionless gait. Blajer and Schiehlen designed and simulated such a controlled collisionless walker that had an asymptotically stable gait [7]. However, the actuators in such devices use energy. Next we discuss tricks to successfully avoid ground collision energy losses.

Kinematic mechanisms. In the late 1800s Chebyshev designed a kinematic one-degree-of-freedom walking device, based on an approximate straight-line mechanism, that guided its endpoints (feet) in a manner that almost avoids ground collisions [8]. A slight change of design could eliminate the small residual collisions [9]. Chebyshev-like mechanisms are used in the beautiful wind-powered walking machines of Theo Jansen [10].

Compliant contact. One could put massless springs on the bottom of the feet. These springs could be compliant in one direction (telescoping along the leg) or in two (2D). If a telescoping leg spring is used, zero-dissipation requires that the foot velocity at contact be parallel to the leg. McGeer found collisionless running motions using such a mechanism [11]. Walking motions with such designs exist, at least using a point-mass body ([12, 13]). If a 2D spring is used, collision losses at first contact are precluded, but release of the contact is dissipative if the tangential spring is still stretched when contact is lost.

Singular-limit of Passive Walking. Garcia *et al.* [3, 14] found that some unactuated mechanisms can walk down arbitrarily small slopes using gravitational power scaling with v^4 (v = average forward speed). Thus these machines use arbitrarily small energy by moving arbitrarily slowly. Indeed, Chatterjee *et al.* [15] proved that such a McGeer-type walker, with no upper body, cannot walk at non-vanishing speeds on vanishing slopes.

In passive-dynamic walkers the normal collision still vanishes as the step-length vanishes. Thus finite-speed vanishing-energy-cost walking machine could use an interleg spring with stiffness tending to ∞, with step length tending to 0, and step frequency tending to ∞ [6, 16].

Our Goal. We search for collisionless motions which don't use the tricks listed above. We seek a collisionless motion of a device that, at least kinematically, allows collisions to occur. That device does not use compliant

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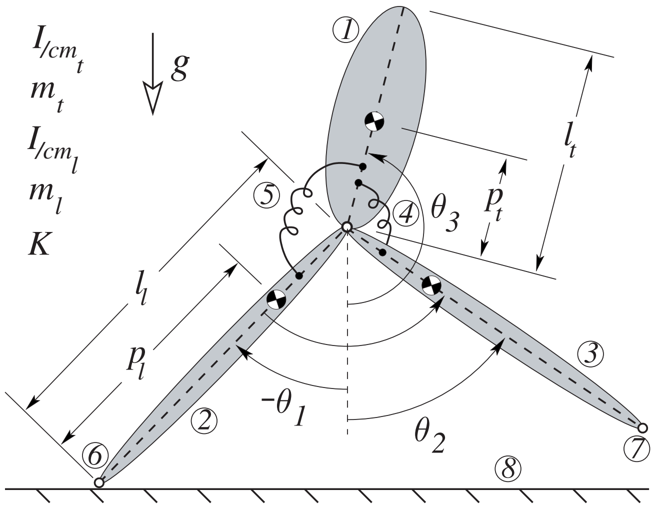

FIG. 1: Three link walking model with hip springs. Modified from [15]. ①torso, ②stance leg, ③swing leg, ④swing leg hip spring, ⑤stance leg hip spring, ⑥stance foot, ⑦swing foot, ⑧ground. Model parameters are: $I_{/cm_t}$=moment of Inertia of the torso about its center of mass, m_t=mass of the torso, $I_{/cm_l}$=moment of Inertia of the leg about its center of mass, m_l=mass of each leg, l_l=leg length, p_t=leg's center of mass location, l_t=torso length, p_t=torso's center of mass location, K= torsional spring constant. Both springs have the same spring constant.

contact with the ground to avoid collisions, and we look for motions that have exactly (rather than in-the-limit) zero energy cost for non-infinitesimal speed walking.

Chatterjee *et al.* [15] conjectured that a passive-dynamic walking device with an upper body (see Fig.1), could walk at a non-infinitesimal speed on level ground. Here we check this conjecture using the model in Fig. 1. Because our model has no actuators and no ground slip, collisions are the only dissipation. Thus, we search for a walking motion of our model that has no collisions as it locomotes across level ground without any energy input.

II. MODEL

The model has two identical legs and a torso connected by a common hinge (Fig. 1 see also [15]). Each leg is also connected to the torso by a torsional spring which is relaxed when the machine stands upright ($\theta_1 = \theta_2 = 0, \theta_3 = \pi$). Thus, if both legs are at rest with both feet on the ground (at any angle ϕ), the upper body has an equilibrium position (stable or not, depending on parameter values) at $\theta_3 = \pi$. We assume two dimensional motion, inelastic (no bounce) ground collisions, and arbitrarily high friction (no slip) at ground contacts.

The equations of motion when one foot is on the ground are determined by using angular momentum balance of the free leg about the hip, of the torso about the hip, and of the whole system about the ground contact point (see Fig. 1). This results in three second order, nonlinear, coupled, ordinary differential equations in the generalized coordinates θ_1, θ_2, and θ_3.

Previously found passive-dynamic walking motions [11, 14, 15, 17–20] assumed one foot on the ground at a time. Because of its upper body, the model here is capable of extended double-stance. For simplicity we only search to motions with no double stance (see appendix B).

For general motions, the discontinuity in $\dot{\boldsymbol{\theta}}$ due to the ground collision would be determined by angular momentum balance of the whole mechanism about the new foot contact point as well as angular momentum balance of the two non-ground-contacting parts about the hip. Because our numerical search was only for collisionless motions (having no velocity discontinuities), we had no need to evaluate the collision-transition relations.

The equations of motion can be rearranged in the form: $[M]\ddot{\boldsymbol{\theta}} = \boldsymbol{v}$, where $[M]$ is a 3×3 matrix depending on state, $\ddot{\boldsymbol{\theta}}$ is a column vector of angular accelerations, and $\boldsymbol{v}$ is a column vector with gravity terms and terms quadratic in the angular rates. Due to the complexity of these equations, we repeatedly formed and solved them numerically as we integrated forward in time (see appendix A).

III. SEARCH FOR ZERO ENERGY-COST WALKING MOTIONS

Given mass and length parameter values and initial values for the state ($\boldsymbol{q} \equiv [\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}]$), the equations of motion determine the subsequent motion. We sought the mathematical result that this system has a periodic solution. The complexity of the governing equations seems to put mathematical proof out of reach, so we sought firm numerical evidence.

As is now common for such periodic locomotion problems ([21]), we treated one walking step as a Poincaré map. Given the state just after one foot fall $\boldsymbol{q}_n$, the solution of the governing equations defines the state after the next foot fall, thus determining a map $F(\boldsymbol{q}_n) = \boldsymbol{q}_{n+1}$. Our goal is to find a fixed point, $\boldsymbol{q}^*$; *i.e.* $F(\boldsymbol{q}^*) = \boldsymbol{q}^*$. This is equivalent to finding a root, $\boldsymbol{q}^*$, of $G(\boldsymbol{q}^*) = \boldsymbol{0}$ where $G(\boldsymbol{q}) \equiv F(\boldsymbol{q}) - \boldsymbol{q}$.

By simple equation counting it seems plausible to find such a root. Finding a root of G is the same as finding a solution of 5 scalar equations in 5 unknowns (the Poincaré section is $6 - 1 = 5$ dimensional). Previous passive-dynamics studies sought, and often found, fixed points of similar maps. We made additional symmetry assumptions to simplify the search.

The model has both temporal and spatial symmetry. Between foot contacts the equations of motion are time reversible but, for general motions, the collision transition equations are not. However, the collisionless solutions we seek have no discontinuities and are thus time reversible; any conjectured solution running backward in time is also a solution. Further, the fore-aft physical symmetry of the device means that any solution can be spatially reflected to obtain a second solution.

We restricted our search to *symmetric* solutions which were unchanged by being simultaneously reflected and time-reversed. Thus half of a walking step fully characterizes the full periodic motion (a full step is shown in Fig. 2).

We used the spatially-symmetric ground contact switching point as an initial condition, now restricted to $\dot{\theta}_1 = \dot{\theta}_2 = 0$ (zero-velocity impact), $\theta_1 = -\theta_2$ (both feet on the ground), and $\theta_3 = \pi$ (symmetry). Defining $\phi = \theta_1 - \theta_2$ at that point results in a 2 dimensional initial condition space $(\phi, \dot{\theta}_3)$. Our target is to have the swing leg straight down and the torso straight up on the map section where the stance leg is vertical $\theta_1 = 0$, $\theta_2 = 0$, and $\theta_3 = \pi$. In other words, we seek special values of the input variables$(\phi, \dot{\theta}_3)$ to the map, H:

$$H(\phi, \dot{\theta}_3)|_{(\dot{\theta}_1 = \dot{\theta}_2 = 0, \theta_1 = -\theta_2, \theta_3 = \pi)} = (\theta_2, \theta_3)|_{(\theta_1 = 0)} \qquad (1)$$

so that the output is $(\theta_2, \theta_3) = (0, \pi)$. The map is $\mathbb{R}^2 \to \mathbb{R}^2$ and our counting argument reduces to $2 - 2 = 0$. Thus for these restricted symmetric solutions it would again be non-degenerate to find isolated solutions for the initial conditions (ϕ and $\dot{\theta}_3$ at $\dot{\theta}_1 = \dot{\theta}_2 = 0, \theta_1 = -\theta_2, \theta_3 = \pi$). We sought a solution to this map by making a guess and doing numerical root finding on Eqn. 1 using Newton's method.

As made plausible by the $2 = 2$ equation counting argument above, we found various fixed-point solutions. Fig. 2 shows our central result, a zero-energy cost walking motion. Although solutions do not exist for all possible mass and length parameter values, solutions seem to exist quite generally. Because we are trying to demonstrate a mathematical result, and not an artifact of numerical approximation, we wanted to find the root as accurately as we could, thus the 12 significant figures given in Fig. 2 (see appendix C).

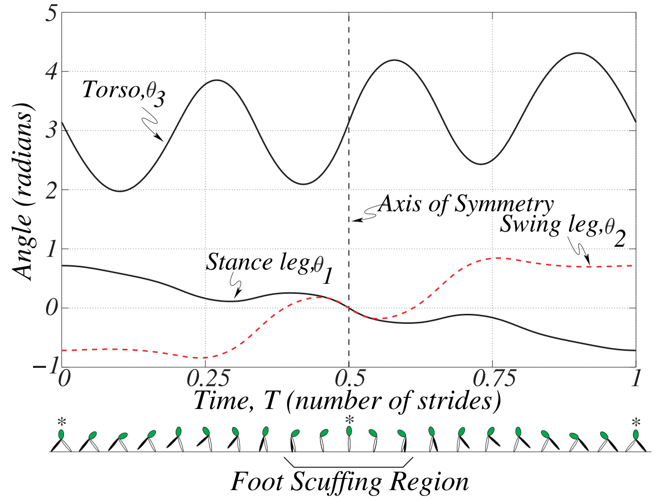

FIG. 2: (Color online)A collisionless periodic solution, without initial ground penetration, for the walking model. The snapshots shown in this figure are spaced evenly in time. The "*" denotes the stride's symmetric points. This solution uses the following non-dimensional parameters: $M_l = m_l/m_l = 1, L_l = l_l/l_l = 1, P_l = p_l/l_l = 0.5, R_l = \sqrt{I_{leg_{cm}}/m_l}/l_l = 0.3, M_t = m_t/m_l = 0.7, L_t = l_t/l_l = 0.6, P_t = p_t/l_l = 0.3, R_t = \sqrt{I_{torso_{cm}}/m_l}/l_l = 0.14, G = g/g = 1, \kappa = K/(m_l l_l g) = 1.5$. The associated initial conditions (the result of the numerical search after convergence tests; see appendix C) are: $\phi/2 = 0.716749728386$, $\theta_3' =$ -7.43120601953, where $(')$ denotes derivatives taken with respect to non-dimensional time, $\tau = \mathrm{t}/\sqrt{l_l/g}$. Despite graphical appearances, θ_1' and θ_2' are not equal at the symmetry point $(T = 0.5)$ $(\theta_1' \approx -2.2$ and $\theta_2' \approx -2.5)$. The non-dimensional period of motion shown above is about 2.621.

IV. RESULTS AND DISCUSSION

We close with a series of observations.

Rigor. This paper claims a mathematical result, the existence of a root of a given function, as inferred by a numerical search. Perhaps one could make the claim mathematically rigorous by using other methods *e.g.* interval arithmetic to generate an interval that necessarily contains a root.

Stability. The stability of the given solution still needs further study. It cannot be formally stable; small perturbations can reduce the energy of the system from which it has no means to recover. However it is possible that the motion is one-sided stable, like the collisionless motions of the hopping block of Chatterjee *et al.* [22], which returns to periodic motion for perturbations which add energy. Alternatively, the solution could also be unstable. However, even though the solution is not asymptotically stable, a device built to implement this motion could probably be stabilized via a controller (with negligible additional energy cost).

Oscillations. An unanticipated aspect of the solution in Fig. 2 is the large number of upper body oscillations per step. We searched for collisionless motions with fewer oscillations, but found that the swing foot would immediately pass down through the ground when weight shift occurred (see appendix B). To avoid this immediate ground penetration, a relatively large angular velocity of the upper body was required. A non-systematic study with approximately 10 parameter sets was done but all single oscillation solutions found violated our ground penetration restriction.

Periodicity. Other zero-energy-cost steady walking motions may exist. The motions we found are symmetric, but we cannot rule out the possible existence of pairs of non-symmetric collisionless motions. We also cannot rule out solutions with a periodicity of more than one step, or even non-periodic solutions. Finally, we cannot exclude the possibility of solutions which have an extended double stance phase.

Applications. We have demonstrated the possibility of using internal oscillations as a means of eliminating collisional dissipation in forward walking. As opposed to all other passive dynamic walking designs to date (*e.g.*

[14, 17, 20, 23–25]), this device walks on *level* ground (in the ideal sense that a wheel rolls steadily on level ground), by eliminating relative velocity at collision. The collision-reduction concept might partially explain aspects of animal and human coordination patterns. For example, Maloiy *et al.* [26] conjecture that some African women developed a technique to achieve a higher than normal efficiency in walking when carrying a load and we surmise it is possible that these women coordinate the motions of their load to reduce collision losses with the ground, thereby reducing metabolic effort. Further, the mechanism shows that there is no fundamental limit to the amount of energy saved by use of orthotics of the type described in [27–29].

Conclusion. This paper shows by means of an example that the only essential energetic costs for walking are those associated with non-ideal effects (joint friction, air friction, imperfections in the trajectories, etc). Energy losses via collisions are not essential for walking, even idealized passive dynamic walking.

Acknowledgments

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APPENDIX A: EQUATIONS OF MOTION AND MAP

We did not explicitly form the equations of motion for this device, as we deemed them too lengthy. Even if generated with computer algebra, we have found that manipulating such large equations adds no insight yet increases chances for error in text manipulation. Rather, we carry out our integration without forming the state derivatives explicitly, generating the numeric values of the derivatives as needed for each integration time step. The use of this procedure makes every line of the program (available upon request) short and interpretable.

There is no possible analytic solution for the two dimensional map. It requires solution for the time, t^*, when $\theta_1(t^*) = 0$. Even for linear problems with closed form solutions, evaluation of this time (t^*) generally involves solution of a transcendental equation [14].

APPENDIX B: GROUND CONTACT ISSUES

Lift off. Upon release of the foot-contact ground constraint, the just released foot can have a positive or negative vertical component of velocity [30]. We excluded solutions where the foot's first motion is through the floor.

Scuffing. Straight legged walkers in 2D will scuff their feet when the swing leg passes by the stance leg. In ours, as well as other straight legged passive-dynamics research, scuffing is ignored. Such scuffing is viewed as a decoupled problem (solved by various means: walking on spaced tiles, retracting ankles, or bending knees). Such anti-scuffing mechanisms used in a physical device will have a small effect on the dynamics.

APPENDIX C: ACCURACY OF INTEGRATION, MAP AND ROOT

We used a fixed step-size 4th-order Runge-Kutta routine to numerically integrate our equations of motion. Henon's method[31] (a change of variables that replaces a traditional root finding procedure) was used to accurately determine the location of the event (mid-stride, $\theta_1 = 0$). Newton's method (using finite differences to approximate the Jacobian) was used to to find the fixed points of the map. The numerical error was estimated in the absence of the true solution by using successive approximate solutions with successively smaller integration step-sizes [32]. At large step sizes truncation errors dominate, at small step sizes round-off errors dominate. A convergence plot shows that round-off error begins to dominate, at a relative error of about 10^{-12}, when the step size is less than about $10^{-4.5}$. Thus our total numerical error is safely below 10^{-11} at that step size.

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