# Prediction of stable walking for a toy that cannot stand

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Previous experiments [M. J. Coleman and A. Ruina, Phys. Rev. Lett. **80**, 3658 (1998)] showed that a gravity-powered toy with no control and that has no statically stable near-standing configurations can walk stably. We show here that a simple rigid-body statically unstable mathematical model based loosely on the physical toy can predict stable limit-cycle walking motions. These calculations add to the repertoire of rigid-body mechanism behaviors as well as further implicating passive dynamics as a possible contributor to stability of animal motions.

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## INTRODUCTION

For walking and other activities, people and animals move in complex, yet stable ways. One view is that such coordination is the action of neuromuscular control constrained by, among other things, the laws of classical mechanics. However, one might ask how much of animal coordination might be understood in purely mechanical terms. Likewise, how much versatility of motion is possible with simple mechanical devices? This paper concerns one example that sheds a little light on these two general questions.

McGeer's (e.g., Ref. [1]) success with straight legged, two dimensional uncontrolled and gravity-powered walking mechanisms highlights the possibility of pure mechanics generating coordination. McGeer found steady walking solutions (periodic gait or limit-cycle motions) that were exponentially stable (asymptotically returned to the periodic motion after small disturbances from that motion). In his two-dimensional theory, only fore-aft stability, and not lateral balance, is in question. In his physical implementations side-to-side balance is enforced by duplicate side-by-side legs (four legs total). These machines cannot stand fully upright, but can stand with splayed legs, possibly contributing to their dynamic stability.

Extending McGeer's ideas, Coleman and Ruina [2] described an easily reproducible [3] two legged gadget built from Tinkertoys® that cannot stand at all, even with both feet on the ground, splayed or not, yet seems (slightly) dynamically stable while walking. But where the stability of McGeer's machines was first predicted with rigid-body modeling, the stability of the Tinkertoy® device was not. As noted in [2], this system is essentially dissipative (from collisions and possibly from ground friction and internal dissipation) and nonholonomic (the dimension of the accessible configuration space is larger than the dimension of the velocity space). Nonholonomic systems can have asymptotic stability, even when conservative, and nonholonomicity from intermittent foot contact might also contribute to stability [4,5].

Which properties are needed for asymptotic dynamic stability of such a statically unstable system was left unanswered by [2]. Possible key factors include friction of the hinge, play in the hinge joint, elastic or inelastic deformation of the structure, compliance at the foot contact, and sliding and twisting friction at the foot contact. Could a rigid-body model without these effects explain the stable motion?

## PREVIOUS RESEARCH

The simulation model in [2] consists of two rigid bodies connected by a frictionless hinge (Fig. 1). The feet are toroidal with principal radii $r_1$ and $r_2$. The ground allows no relative motion of contacting points, no torques at the foot contact, no bounce (restitution $e=0$), and no tension force at the foot contact (nonsticky floor). The lengths, center of mass location, the moment of inertia components, and the ground slope are adjustable. After nondimensionalizing with mass $m$, length $l$, and time $\sqrt{l/g}$ there are 13 free parameters.

The working Tinkertoy® was based on mildly unstable simulations [2] of a simplified model with point contact ($r_1$

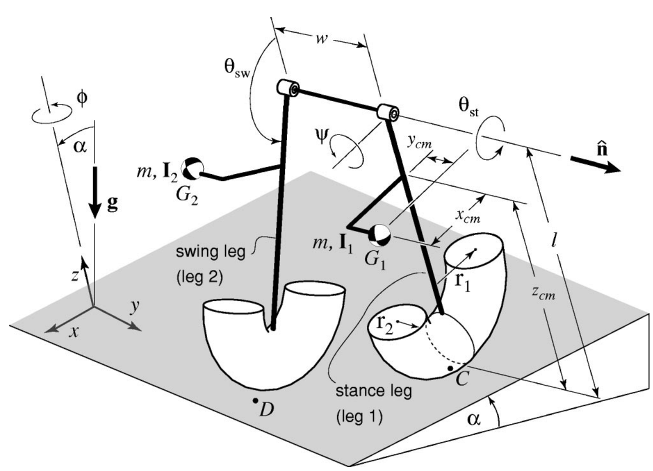


FIG. 1. The 3D rigid-body model. The parameters and state variables are described in [2].

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$=r_2=0$) and narrow hips ($w=0$).

Earlier, McGeer [6] studied the same model, allowing $r_1>0$ and $w>0$ but assuming that the principal moments of inertia aligned with the hip hinge and leg. He found only unstable solutions where, also, the swing foot passed below ground. Kuo [7] studied a similar model, but disallowed steer ($\phi$) and also only found unstable passive gaits. Dankowicz *et al.* [8] found stable solutions for a related kneed computational model. That model has wide feet so, like the 2D models, it can stand stably with splayed legs. The semi-3D computational model of Wisse *et al.* [9] can also stand with splayed legs.

## METHODS

Our study was of the system in (Fig. 1) [2] described above, but with hip spacing and disc feet ($r_1\neq 0$, $w\neq 0$, $r_2=0$).

The overall approach is to characterize the solution of the rigid-body dynamics equations as a function (map) that takes the state of the system just after one step as input and gives the state just after the next step as output. A fixed point of this map defines a limit cycle. Stability is evaluated by the eigenvalues of the matrix describing the linearization of this map. If all eigenvalues are less than 1 in modulus the periodic motion is exponentially stable. The map, its fixed points and the linearization are all found from numerical solutions of the equations of classical rigid-body dynamics.

The numerical searches were aimed at generating stable motion and not at accurately modeling either the existent physical toy or humans. We used the toy's approximate parameters to seed the optimizations. Special purpose optimal control software (see Appendix) was used to reduce the most unstable eigenvalue while maintaining periodicity of the gait, positive foot clearance, and static instability. The resulting solution was checked and improved with an independent method and checked again with another independent simulation.

## RESULTS

The model of Fig. 1 has asymptotically stable limit-cycle motions (Fig. 2), with the foot of the swing leg clearing the ground, with $I_{XX}=0.1982$, $I_{YY}=0.0186$, $I_{ZZ}=0.1802$, $I_{XY}=0.0071$, $I_{XZ}=-0.0023$, $I_{YZ}=0.0573$, $\alpha=0.0702$, $X_{cm}=0$, $Y_{cm}=0.6969$, and $Z_{cm}=0.3137$, $W=0.3624$, and $R_1=0.1236$ and $R_2=0$. Capital letters indicate non-dimensional variables. Tensor components $I_{MN}$ and mass positions are in local left-leg coordinates with origin at the vertically standing contact. Note the static instability ($Z_{cm}>R_1$). The largest eigenvalue modulus of the single-step map Jacobian is 0.839 156 0, safely below 1.

## DISCUSSION

We are claiming a qualitative theoretical mechanics result. That is, a system described with the classic equations of rigid-body mechanics has an exponentially stable limit-cycle solution in the neighborhood of a statically unstable configuration, but with no fast spinning parts. Although we have not attempted a mathematical proof, we have attempted to do our numerics with enough checks and tests to state the result with confidence (see Appendix below).

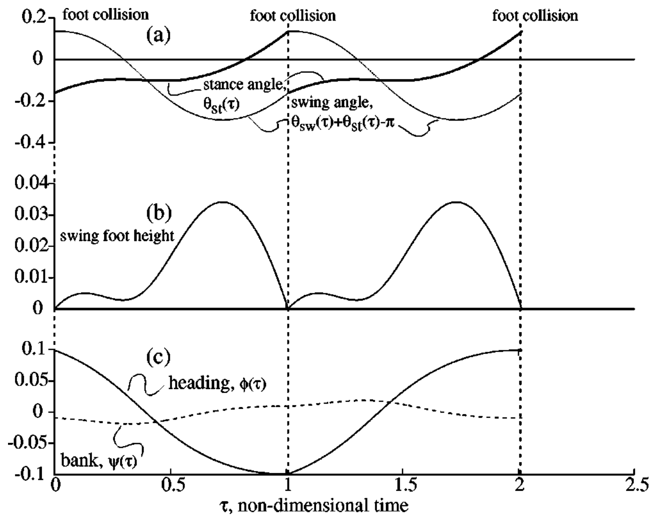


FIG. 2. A gait cycle (two steps). In the first half, the stance leg is the left leg. (a) The swing leg angle is here measured from the slope normal $[\theta^*_{sw}(\tau)+\theta^*_{st}(\tau)-\pi]$; (b) positive swing-foot clearance between collisions; (c) the motion has more steer (yaw) than lean (bank).

This solution can be interpreted as bipedal walking, although not especially anthropomorphic. The base solution is exactly repetitive, step after step. That the largest eigenvalue is less than 1 in magnitude, means that, if the mechanism were slightly disrupted from this periodic motion it would asymptotically approach this motion again, over a number of steps.

Simple numerical probes show, as exponential stability demands, a small but noninfinitesimal basin of attraction. We have not investigated the shape or size of this domain in detail; we do not know exactly what set of motions eventually settle into the periodic motion and thus cannot precisely describe what disturbances knock the walker down. However, the success of the physical model [2] suggests that the stability is robust enough to be physically relevant.

We do not claim to have an accurate model of the toy in [2]. Rather we have a simple model that explains the toy's qualitative behavior. Accurate quantitative prediction of the toy's motions may well depend on physical effects that are not in our simple model (various frictions and deformations). Yet unknown is whether the parameters presented here could be used as a basis for a better working physical device. More generally, we also do not know if more humanlike stable passive-dynamic designs can be made that are also statically unstable.

## CONCLUSIONS

The dynamic stability of a statically unstable walking mechanism can be predicted with a model consisting of two rigid bodies connected by an ideal hinge and making intermittent ideal no-slip, no-bounce point contact with the ground. We have shown that there is no need to appeal to hinge-friction, hinge-play, distributed or contact deformation (elastic or inelastic), or contact frictional slip in order to qualitatively predict the interesting behavior demonstrated by the physical model in Ref. [2].

The results further highlight the versatility of simple passive strategies for stabilization of coordination. The calculation also slightly expands the range of known rigid-body phenomenology.

For videos and reprints about the Tinkertoy® and related machines, see www.tam.cornell.edu/~ruina/hplab/pdw.html.

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## APPENDIX: NUMERICAL ANALYSIS

We carried out the numerical analysis in three different ways.

The first stable solution (with $|\sigma_{max}|=0.897$) with foot clearance was found using the approach developed by Mombaur *et al.* [10] on the basis of the optimal control code MUSCOD by Bock *et al.* [11] and Leineweber [12]. In the language of the discipline, MUSCOD has been written for general multiphase optimal control problems and is based on a multiple shooting state discretization. Multiple shooting splits up the original boundary value problem into a number of initial value problems enforcing continuity conditions at the transitions from one interval to the next. At all multiple shooting points MUSCOD allows the user to impose a number of equality and inequality constraints on the parameters and the dynamic variables being varied in the optimization. For the Tinkertoy® model described here, this permitted us to ensure periodicity, foot clearance during the step, and to keep all state variables and parameters within reasonable ranges. Sensitivities of the integration end values on each interval, both to variations in initial values and to variations in model parameters, are efficiently computed by means of internal numerical differentiation (IND). The basic principle of IND is to use identical, but adaptive and error-controlled discretization schemes for integration and derivative generation. For use with actuated and passive gait problems, with implicit state-dependent phase switching points and discontinuities in the state variables, the original MUSCOD has been combined with an object oriented modeling library that deals with these situations in a uniform and complete way. Also added to MUSCOD were stability analysis modules that compute the linearized Poincaré map of a periodic solution and their eigenvalues, assembling information from all multiple shooting intervals and taking care of the above mentioned implicit switching point dependencies. Stable solutions for the Tinkertoy® model were found by varying model parameters and bounds based on coarse grid sensitivity information gathered during the previous optimal control problem solutions.

Second, we reproduced and improved the solution above by fourth order Runge-Kutta integration of the governing ordinary differential equations in Matlab®, finding the collision time accurately using Henon's method (changing the independent variable near the collision time). The impact transition is a matrix multiplication. The fixed points of the resulting return map were found by numerical root finding. The fixed point map Jacobian was found by a central difference perturbation of the initial state. The eigenvalue was reduced from 0.897 to 0.839 using a simulated annealing optimization of the maximum eigenvalue modulus. For this MATLAB solution we did extensive convergence tests on both the integration step size and the central difference step size. These tests indicate a combined roundoff and truncation error of about $\pm 10^{-7}$ in the largest eigenvalue modulus for the parameters given. This maximum eigenvalue estimate differs from that generated by MUSCOD with these parameters by $2\times 10^{-3}$.

Finally, the equations of motion were derived independently and simulated again independently in MATLAB giving agreement to the MATLAB solution above $10^{-6}$ for the largest eigenvalue modulus.

For reference, the state of the system just after collision of the left foot is, for the parameters given, $\mathbf{q}^*=[\phi,\psi,\theta_{st},\theta_{sw},\dot{\phi},\dot{\psi},\dot{\theta}_{st},\dot{\theta}_{sw}]=[0.098\,68, -0.009\,25, -0.160\,16, 3.435\,83, -0.132\,21, -0.019\,91, 0.471\,24, -0.392\,56]$ with step period $\tau^*=1.007\,11$ where $(\dot{\ })=d(\ )/d\tau$ with $\tau$ the dimensionless time.

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