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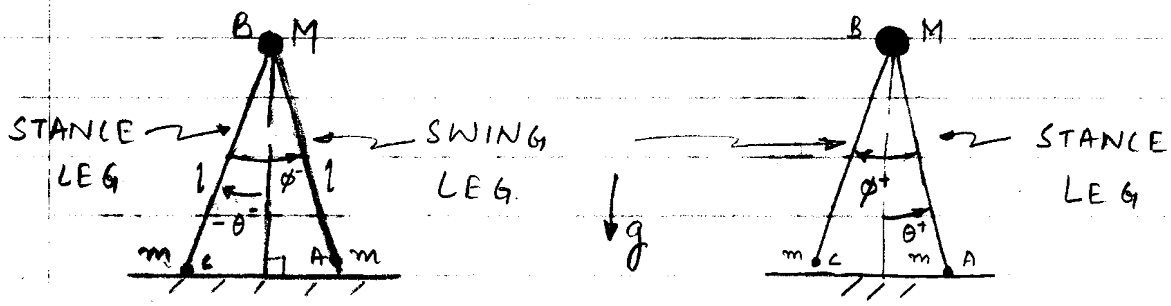
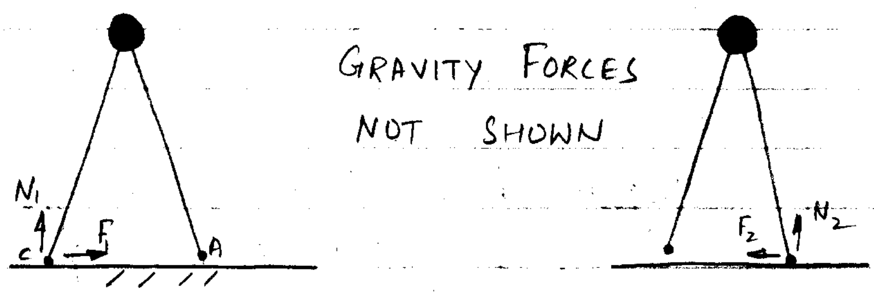
The Simplest Walking Model: Stability, Complexity and Scaling.

Derivation of the transition rule at heelstrike:



Just Before Heel Strike

Just After Heel Strike



To get the transition rule we will conserve ANGULAR MOMENTUM of the system about the point on the ground and coincident with A.

From the figure it can be seen that the only force giving rise to an angular impulse during the time Δt (during which the heelstrike occurs) is N_1. This is because N_2 and F_2 are acting at A while F_1 passes through A.

However, we will assume that the angular impulse due to N_1 is negligible as compared to the ones due to the forces at A. We can attempt to justify this by the fact that C is "LIFTING OFF" as opposed to A which is "STRIKING".

Making the above assumption enables us to equate the angular momentum of the system before and after heelstrike.

Now,

Angular Momentum of the system
= Angular Momentum of mass at A
+ that of mass at B + that of C.

We further assume that the hip mass, M is much greater than the leg mass m, i.e

$$M >> m$$

Thus, the angular momentum of C can be neglected in comparison to B's. As the angular momentum is being taken about A (on the ground), angular momentum of A is identically zero.

Hence,

$$M\, V_B^-\, r_{BA\perp}^- = M\, V_B^+\, r_{BA\perp}^+ \qquad \text{— (1)}$$

where,

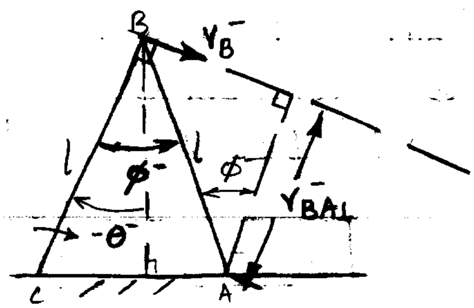
V_B^- : Velocity of B before heelstrike

V_B^+ : Velocity of B after heelstrike

$r_{BA\perp}^-$: distance of A from the line of action of V_B^-

$r_{BA\perp}^+$: distance of A from the line of action of V_B^+.

To obtain V_B^-, $r_{BA\perp}^-$ consider:



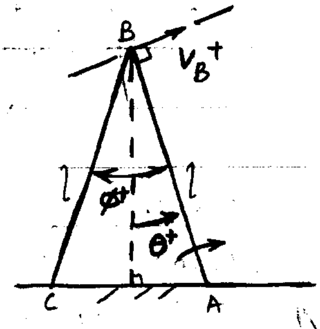
Before heelstrike BC is rotating about C with angular velocity $= \frac{d}{dt}(-\theta^-)$

$= -\dot{\theta}^-$

hence,

$$\left.\begin{aligned} V_B^- &= -l\,\dot{\theta}^- \\ r_{BA\perp}^- &= l\cos\phi^- \end{aligned}\right] \qquad \text{— (2)}$$

For V_B^+, and $r_{BA\perp}^+$,



After heelstrike AB starts rotating about A with an angular velocity $= -\dot{\theta}^+$. Note the negative sign, it has been used because decreasing θ^+ corresponds to the direction of V_B^+ shown in the fig.

thus,

$$\left.\begin{aligned} V_B^+ &= -l\dot{\theta}^+ \\ r_{BA\perp}^+ &= l \end{aligned}\right] \quad — \quad (3)$$

Substituting (2) and (3) in (1)

$$-Ml\dot{\theta}^- l\cos\phi^- = -Ml\dot{\theta}^+ l$$

or, $$\boxed{\dot{\theta}^+ = \dot{\theta}^- \cos\phi^-} \quad — \quad (4)$$

To complete **the process** we isolate BC and write the conservation of angular momentum equation about a point coincident with B but fixed in space.

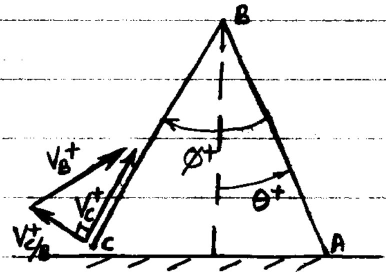
Angular impulse due to contact forces at C are again neglected.

Angular momentum of M about C will be zero. And so we will get,

$$m V_c^- r_{CB\perp}^- = m V_c^+ r_{CB\perp}^+ \quad — ⑤$$

where the terms have meanings similar to the terms in eq^n ①.

Now, before heelstrike $V_c^- = 0$.
hence the LHS of ⑤ is zero.
After heelstrike,



The figure also shows the velocity diagram of point C. It graphically represents the equation

$$\underline{V_c}^+ = \underline{V_B}^+ + V_{c/B}^+$$

where $V_{c/B}^+$ is the velocity of C w.r.t B

Note that the only way ⑤ can be satisfied is when $r_{CB\perp}^+ = 0$.

This in turn means that the line of action of V_c^+ passes through C.

So,

$$V_c^+ = V_B^+ \cos\left(\frac{\pi}{2} - \phi^+\right)$$

from ③ $\quad V_c^+ = l\,\dot{\theta}^+ \sin\phi^+ \quad — ⑥$

Also,

$$V_c^+ = V_{c/B}^+ \cot\left(\frac{\pi}{2} - \phi^+\right)$$

$$= l(\dot{\phi}^+ - \dot{\theta}^+) \tan\phi^+ \quad \text{— (7)}$$

Using (6) in (7) and rearranging

$$\boxed{\dot{\phi}^+ = (1 + \cos\phi^+)\,\dot{\theta}^+} \quad \text{— (8)}$$

Now for some geometrical considerations,

Noting that the transition at heelstrike is practically instantaneous we can write down the following equations,

$$\left.\begin{array}{l} \phi^+ = -\phi^- = \phi \\ \theta^+ = -\theta^- = \theta \end{array}\right] \quad \text{— (9)}$$

And since the length of the legs is equal,

$$\phi = 2\theta \quad \text{— (10)}$$

(4) and (8) together give

$$\boxed{\dot{\phi}^+ = (1 + \cos\phi^+) \cos\phi^- \,\dot{\theta}^-} \quad \text{— (11)}$$

and (4) and (11) taken together with (9) and (10) give,

$$\boxed{\begin{aligned} \dot{\theta}^{+} &= \cos(2\theta)\ \dot{\theta}^{-} \\ \dot{\phi}^{+} &= \cos(2\theta)\left[1+\cos(2\theta)\right]\ \dot{\theta}^{-} \end{aligned}} \quad \text{—— (12)}$$

Now we can put (9), (10) and (12) in matrix form to get the desired transition rule,

$$\begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix}^{+} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \cos(2\theta) & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & \cos(2\theta)\left[1-\cos(2\theta)\right] & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix}^{-}$$