Two Interpretations of Rigidity in Rigid Body Collisions

Anindya Chatterjee
anindya@crash.esm.psu.edu
Department of Engineering Science & Mechanics
The Pennsylvania State University
University Park, Pennsylvania 16802

Andy L. Ruina
ruina@cornell.edu
Theoretical and Applied Mechanics
Cornell University
Ithaca, NY 14853

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Abstract

We distinguish between, and discuss the applicability of, two levels of rigidity in rigid-body collision modeling. For rigidity in the strong, *force-response*, sense collisional contact deformations must be highly localized. The bodies then move according to second order rigid-body mechanics during the collision. Incremental collision laws and most collision models using continuum mechanics for the contact region depend on force-response rigidity. For rigidity in the weaker, *impulse-response*, sense the deformations need not be localized but displacements during the collision need to be small everywhere. Only the time-integrated rigid-body equations, involving before-collision and after-collision velocities, then need apply. Although a force-response rigid body is also impulse-response rigid the converse is not true. Algebraic collision laws depend only on impulse-response rigidity. Elastic vibration models of collisions are also generally consistent with impulse-response rigidity.

1 Introduction

Modeling collisions, the brief strong contact interactions of solid bodies, is of interest both for applications including robotics and general multibody dynamics simulations and to help complete classical mechanics, which does not include a settled approach to constitutive laws for contact. In dynamic models of mechanical systems, an extremely popular and useful idealization of a solid object is as a *rigid body.* The world of ideal rigid bodies has clear and well defined rules for how objects move under the action of forces and moments, as well as how ideal constraints like perfect rolling, frictionless sliding, and frictionless joints restrict the motions of systems of objects. To complete the model-world of ideal rigid bodies, rules are required for frictional contact, dissipative rolling, and (the topic of this paper) collisional contact.

The effect of collisional contact on motion is determined by the momentum balance equations and collisional constitutive relations (also called collision laws). But many questions are still not settled: how general, sensible collision laws might be constructed; how good the basis is for any existing law; and how useful the predictions are of these or any other laws in this world of not-truly-rigid bodies.

One set of important questions concerns the role of rigidity in rigid body collision mechanics. We aim here to help clarify the applicability of two different notions of rigidity in collisions.

2 Deformations and displacements during collisions

A collision of two solid objects is shown schematically in Fig. 1. There is a small region of intense deformation near the initial contact point C. Outside but near this region is a reference point R. A typical point in the body is shown as P. The issue at hand is to what extent the points C, R, and P in one body may be viewed as rigidly linked.

Since the contact forces of colliding, nominally non-deforming, 'rigid' bodies are governed by their deformations, some confusion naturally follows from the oxymoronic phrase 'rigid-body collisions.' One approach to resolving the resulting ambiguities is to consider a sequence of approximating systems having high stiffness and/or damping (see Brogliato (1996) and references therein). The limiting case of infinite stiffness and/or damping gives a possible realization of a 'rigid' object, and a way to develop rules for the behavior of such rigid objects.

A slightly different view is that rigidity itself is an approximate description of real bodies that are suitably stiff. In this view, the only well defined aspects of rigid body mechanics are those aspects of deformable body mechanics that do not depend on how limits of infinite stiffness and/or viscosity are approached. Features of the mechanics of deformable bodies that remain dependent on the details of the deformations, even as the deformations vanish with increasing stiffness, need further resolution by the introduction of additional constitutive assumptions beyond just 'rigidity.' In particular, 'rigid body' collision models are completed by introducing constitutive assumptions for the net collisional outcome, for contact deformations, or for whole-body deformation. Despite the implied deformations in each case, some notion of rigidity can still be retained.

Consider for example the Hertzian contact between two frictionless, elastic spheres pressed together with remotely applied, equal and opposite forces (possibly including smooth D'Alembert inertial terms) of magnitude F (for a detailed solution, see Johnson, 1985). Important features of such contact interaction are: (i) the contact region is small compared to the overall dimensions of the contacting bodies, and (ii) the contact stresses σ_{ij}, and the associated deformations, decay rapidly at large distances from the contact region. For spheres and other non-conforming 3D elastic contact problems stresses decay like $1/r^2$, where r is the distance from the contact region. Large portions of the contacting spheres move with relatively smaller deformation. As a result, the net relative displacement $\delta = \Delta\mathbf{u}$ between typical points

P, say the centers of the spheres, can be accurately calculated *without* knowledge of exactly where and how the remote forces (i.e., F) are applied on the spheres. For 3D contact that is sufficiently smooth in time, the relative displacement between pairs of points that are not in the contact-deformation regions (say points R and P) can be accurately calculated by treating the spheres as rigid everywhere except in a small, localized contact region governed by the Hertz relations.

In general point-contact problems, similar qualitative features are expected to that of the sphere for a useful range of material properties and of objects. The key ideas are that **(a)** the contact stresses are large and decay rapidly with distance from the contact region; **(b)** the contact region stresses and deformations depend only on the net resultant contact force (and possibly its history – see e.g. Mindlin and Deresiewicz (1953)); **(c)** there is an intermediate-sized region inside which the interaction is pseudostatic ($\sigma_{ij,j} \gg \rho\ddot{u}_i$, where ρ is density), and outside which the object has small deformations; and **(d)** the true contact region and intermediate size region are much smaller than the colliding objects. The natural rigid approximation is that the connection between P and R is rigid and all deformation between R and C is pseudo-static.

Consider next the longitudinal contact of long bars. In this case the deformation in the bulk of the rods (i.e., terms like FL/AE where F is the force, E is Young's modulus, L is length and A is cross sectional area) will usually dominate any contact deformation and the motions between points on the two rods can be well approximated without including any extra contact compliance. That is, the region from C to R may be represented as arbitrarily small and rigid and all deformation is between R and P.

Finally, in the intermediate case of Hertzian contact between cylinders in plane strain or disks in plane stress, the stress field decays slowly enough (like $1/r$) that the deformations are not localized near the contact region. The net relative displacement between the centers of the cylinders, say, cannot be determined without consideration of the global deformations characterized by the relative displacement of R and P (even though the strains between R and P are small compared to unity).

In the case of spheres the deformation between R and P can be neglected. In the case of longitudinal contact of long rods the deformation between R and C can be neglected. And in the case of disks and cylinders both contact and far field deformations are important for determining motions of general points on the body. Yet in all three cases the relative displacements of typical points can be far less than the distance between them.

Motivated by the observations above, we examine qualitative features of collisions where the effects of localized contact deformations dominate over global deformations, and vice versa.

3 The contact point and contact region deformation

We discuss here only single-point contact collisions where the material contact point C is the point of first contact of two bodies that are locally relatively convex, or non-conforming (see Fig. 1). Some of the ideas here are relevant to simultaneous collisions at multiple points [1] although, as explained clearly by Ivanov (1995), 'simultaneous' collision behavior is often severely ill-posed. (Ivanov's paper refers to some early works in this area, including D'Alembert's (1743) demonstration that a detailed solution incorporating elastic deformations leads to a prediction different from considering a sequence of pairwise collisions.)

One might, unlike what is shown in Fig. 1, take the reference contact point R to be the position in space that the material point C would have if not for local contact stresses. For example, if the body is effectively rigid but for localized contact deformations, then point R might be chosen as the point in space that is the same distance from points interior to the body as was material point C before the

[1] Attempts to model multiple simultaneous collisions have been made by many authors, e.g., Pfeiffer and Glocker (1996).

collision[2].

Since we assume *a priori* that the region of contact is small compared to the size of the body, impulse momentum relations are only slightly affected by which point inside or near it is used for the application of the collisional impulse. Consider a collision where a net impulse of magnitude P acts at a point C on a body of characteristic length L. The moment of the impulse about the body center of mass G typically has a magnitude comparable to PL. If the impulse were taken to act at a point a small distance δ from C, the correction to the impulsive moment about the center of mass is comparable to $P\delta$, which is negligible if $\delta \ll L$.

In the degenerate case where the impulse $\mathbf{P}$ is parallel to CG, such as when a uniform sphere is dropped vertically onto a horizontal floor, possibly with spin about a vertical axis, a term of magnitude $P\delta$ may not be small compared to the nominal impulsive moment about G (Brach (1991) attributes this example to Horak). If terms of order $P\delta$ are deemed important then either the collision is not strictly treated as a point collision, or for consistency impulsive moments about the contact point need be considered. Here we ignore such terms because they are small, as described, for single-point collisions of stiff bodies.

So, with little loss of accuracy, the reference point R can be taken as any material point in the body that is slightly removed from the local contact region. One might think of R as representative of a point in the region surrounding the contact region that separates the series connection of contact and bulk deformations.

In any case, our definition of contact-region deformation is the relative displacement of the reference points on the two bodies (say R_1 and R_2) during the collision. If one of the bodies, say body 2, deforms much less than the other, then its initial contact point C_2 may be used for its reference point R_2. For this reason, a collision with a rigid wall serves our explanatory purposes and henceforth we take the contact region deformation as occuring between points R and C on one body.

Following the treatments of 'approximating problems' by Brogliato (1996) and references therein, we illustrate the ideas here with the one-dimensional cartoon shown in Fig. 3b as well as special cases shown in Figs. 4a-c. Figure 3b shows the cartoon idealization of a solid body. The contact point C and a nearby reference point R are separated by a massless contact deformation mechanism. The point R and point P are separated by the small-deformation mechanics of the bulk of the body.

4 Ideal rigidity is well defined for smooth motions

In studying the dynamics of a 'real' object as idealized in Fig. 3a the simplifying assumption of rigidity is that the change in distance between any pair of points on the body is zero for all time. Under the assumption of rigidity the velocity of *any* material point P relative to the material point C is given by the cross product of the object's angular velocity vector and the position vector from C to P, i.e., $\mathbf{v}_{\mathrm{P/C}} = \boldsymbol{\omega} \times \mathbf{r}_{\mathrm{P/C}}$. In 1D rigidity, this reduces to $v_{\mathrm{P/C}} = 0$.

Consider the cartoon in Fig. 4a. If a smooth, bounded force $F(t)$ is applied at point C, then the acceleration of the block, i.e. point R (or P), is F/m. The acceleration of point C is *not* F/m, due to the deformation of the spring. However, the limiting case of infinitely stiff spring and dashpot is well-defined. The motions of both points, C and R, may be treated as identical if we ignore some vanishingly small-amplitude phenomena. In energy calculations, the work done by the force will be almost totally converted to kinetic energy of the block, with a vanishingly small portion dissipated in the dashpot. In this sense, for smooth motions under bounded forces, 'infinitely stiff' is a sufficient

[2]The extrapolation of the position of a material point based on neglecting local contact deformations is not always well-defined, e.g., for two dimensional elastic objects with Hertzian contact. We do not know of a rigorous argument that delineates when such a point R is or is not well defined in some suitable asymptotic sense.

criterion for describing 'ideal' rigid behavior. We need not examine the relative rates at which k and c approach infinity, i.e., quantities like the limiting value of $c/\sqrt{mk}$.

A similar analysis of the model in Fig. 4c, and hence for the model in Fig. 3b, also leads to an unambiguous infinitely-stiff limit for smooth forces. Another analysis for the case of smoothly varying boundary motions (i.e., imposed velocities at boundary points) also shows a unique rigid limit.

For smoothly applied forces, or for smooth displacement boundary conditions, 'rigid' may generally and unambiguously be taken to mean 'no deformations.' This ideal rigidity, where points C, R, and P keep fixed distances from each other, is a well posed limit of high stiffness.

5 Ideal rigidity is not well defined for collisions

Now consider the system in Fig. 4a colliding with a rigid wall as in Fig. 4b. In this case, there is a period of compression at the end of which the block is momentarily at rest, followed by a period of 'restitution' or rebound during which the block accelerates away from the wall. At the end of compression, all the energy of the block has been absorbed by the spring/dashpot. Some of it subsequently returns to the block, while the rest is dissipated. If we now make the spring and dashpot stiffer, the interaction with the wall occurs over a shorter time. In the limiting case of $k \to \infty$, we have an instantaneous[3] 'collision.' However, the entire energy of the block still goes into the spring and dashpot at the end of the infinitely short compression phase, and what fraction of it is returned depends on the limiting value of $c/\sqrt{mk}$ (see, e.g., Chatterjee (1997) or Brogliato (1996) for slightly different treatments). Thus, there is no single, unambiguous 'ideal rigid' limit for this model and hence for solid bodies in general.

In the case of collisions the limit of infinite stiffness – for which the distances between points C, R, and P stay fixed – does not lead to a unique outcome. That is, various degrees of rebound, energy dissipation, etc., are possible in the infinitely stiff limit. To predict such quantities, we need to make additional hypotheses about the collisional interaction.

6 Basic assumptions

We assume, as do most authors, several features which seem well approximated by many collisional interactions of interest. All contact forces are in a small contact region; before and some time soon after the collision the velocities of points in the bodies are described with rigid-body dynamics relations; the configuration of the bodies does not change appreciably during the collision (i.e., the displacements of points in the bodies are small compared to the sizes of the bodies); and during collision the collisional forces are much bigger than other forces or $O(\omega^2)$ inertial forces.

These assumptions imply that the net change in linear momentum, angular momentum, and energy of the bodies can be determined by their pre- and post-collision rigid-body velocities and angular velocities. Also, the contact interaction is summarized by the net resultant contact impulse, equally and oppositely applied to the contacting bodies at their contact points.

7 What is a rigid body collision law?

Given a pair of colliding rigid bodies (or mechanisms composed of rigid bodies connected with frictionless geometric constraints) in known configurations and with known pre-collision velocities, and given the

[3]If the limiting value of k/c is finite, the interaction time does not actually go to zero. However, in this case deformations are infinitesimal, and only some infinitesimal restitution occurs over finite time.

relevant parameters (which could be coefficients of friction, restitution, material properties, geometric information like local surface shape, etc.) a *collision law* predicts the contact impulse.

Linear and angular momentum balance relations are not part of the collision law but, given the contact impulse, do uniquely determine the post-collision velocities.

If instead of impulses one wants to express the collision law directly in terms of the motions of the bodies, then these motions must be chosen so as to satisfy the basic momentum relations.

8 Definitions of rigidity in collisions

8.1 Force-response rigidity

We refer to an object as effectively *force-response rigid* in a given collision if **(a)** a well-defined pseudo-static contact region (R to C) can be identified, and **(b)** bulk deformations between R and P are negligible at all times including *during* the collision. Force-response rigidity is the strongest sense in which a body may be considered as rigid during a collision. Away from the contact region, such an object behaves like an ideal rigid body moving under the action of a concentrated force acting at the contact point.

Since the collisional contact force is dominant, the dynamic behavior of a force-response rigid object is summarized by the linear relation between the contact force vector $\mathbf{F}$ and the acceleration $\mathbf{a}_\mathrm{R}$ of the near-contact reference point

$$\mathbf{F} = \mathbf{M}\mathbf{a}_\mathrm{R} \tag{1}$$

where $\mathbf{M}$ is a second order tensor with units of mass, that includes effects of mass, rotational inertia and kinematic constraints, and that expresses the net anisotropic inertia encountered by a force acting at that point[4] (this inertia tensor or 'mass matrix' is well known – see e.g., Batlle (1993), Smith and Liu (1992), Chatterjee (1997)). Given that real bodies do deform, the approximation being made in force-response rigidity is that the contact region relative displacements (between C and R) are much greater than any deformation-induced relative displacements between points removed from that region (e.g., R and P).

In principle one can test the accuracy of the force-response rigidity assumption. First assume the body is force-response rigid and calculate (using the micro-mechanics of the contact region, the initial velocities, and the mass tensor $\mathbf{M}$) the time history of the net contact force. Then apply this force to a more realistic model of the body as a whole to check that the deformation-induced relative displacements between R and P in this second calculation are much smaller than those between C and R. In our 1D cartoon this test corresponds to first using the model in Fig. 4b and then using the force history thus found on the model in Fig. 4c and checking that the R to P relative displacement in the second calculation is relatively much smaller than the C to R relative displacement.

If a body is well described as force-response rigid the net collision impulse can only be a function of material behavior in the contact deformation region, the relative velocities of the contact points before collision, and the mass tensor $\mathbf{M}$. That is, the collision impulse cannot depend on any aspects of the shape or mass distribution of the object or mechanism that are not encompassed in $\mathbf{M}$ (see Chatterjee, 1997). Collision laws based on contact micro-mechanics mediating the interaction of otherwise rigid bodies (*incremental* collision laws) are based on the implicit assumption of force-response rigidity.

Numerical Example: For illustration, assume that the contact force history from the model in Fig. 4b is a constant force F over a time interval Δt with a net impulse $P = F\Delta t$. We now examine the behavior of the system in Fig. 4c for large stiffness and damping. In Fig. 5, the velocity of the

[4] The nonlinear ω^2 terms vanish in proportion to other inertial terms in the limit as the interaction time goes to zero and the interaction force goes to infinity.

mass m_1 is shown as a function of time. It is seen that for increasing k and c (for simplicity, holding $c/\sqrt{km}$ constant), the response of the system approaches that predicted by force-response rigidity. The reference point R on mass 1 follows the second order Newton-Euler dynamics relations based on the entire mass of the system: $F \approx ma_1$ applies (where a_1 is the acceleration of block 1). The common description of this situation is that the settling time for vibrations of the body as a whole are much less than the contact interaction time.

8.2 Impulse-response rigidity

We refer to an object as effectively *impulse-response rigid* in a given collision if **(a)** displacements are small compared to the dimensions of the object, everywhere and at all times during the collision, and **(b)** the object has well-defined rigid-body motions before and after the (brief) collision.

For such an object, integration of the deformable body equations of motion show that the net impulse vector transmitted in the collision is related to the net change in the velocity of the near-contact reference point, vector $\mathbf{v}_\mathrm{R}$, by an impulse-momentum relation of the form

$$\mathbf{P} = \mathbf{M}\Delta\mathbf{v}_\mathrm{R}, \tag{2}$$

where $\mathbf{M}$ is the same second order tensor with units of mass, described earlier, that would be used for the bodies were they force-response rigid. In particle mechanics the momentum equation $P = M\Delta v$ is derived from $F = ma$. For the problem at hand, however, the integrated form of the rigid body momentum equations Eq. 2 is accurate even for cases where the rigid body differential equation Eq. 1 is inaccurate.

An object is rigid in the weaker impulse-response sense if it moves enough like a rigid body before the collision and soon enough after that its momentum, angular momentum, energy and the velocity of point R can be calculated accurately using rigid body relations with the same configuration (position and orientation) after the collision as before. Impulse-response rigidity applies to a wider range of collisions than force-response rigidity.

In the cartoon model there is a range of k and c for which the transients decay on a time scale comparable to Δt. In this range of stiffness, the system is rigid in the weak (impulse-response) sense – on time scales that are long compared to the time scale of the impulse, *before* and *after* but *not during* the application of the impulse, the system moves essentially like a rigid body (see the 'weak-rigidity' curve in Fig. 5). The net difference between the pre- and post-collision rigid-body-like motions is accurately described by rigid body impulse-momentum relations. In other words, the collisional behavior of the object is well-described by $P = m\Delta v_1$ (where v_1 is the velocity of block 1) even though the equation $F = ma_1$ is inaccurate. After application of the impulse P and after transients die out, the velocity of each block is equal to $P/(m_1 + m_2)$, so the *net* change in velocity *can* be found by approximating the system as a single rigid object.

If a body is only impulse-response rigid there is no reason to expect that collision laws based solely on the local deformation mechanism, the incoming velocity difference, and the mass tensor $\mathbf{M}$ should be accurate. Aspects of the body not encompassed in $\mathbf{M}$ are likely to be important determinants of the collisional impulse. But it is still true that the net jumps in velocities of all points is determined by the collisional impulse and that the mass tensor $\mathbf{M}$ helps in the expression of reasonable bounds on possible collisional impulses (Chatterjee and Ruina (1998)).

8.3 Non-rigidity

Finally we consider a case where rigidity does not even apply in the weak impulse-response sense.

If the spring and dashpot in Fig. 4c are soft, so that the time scale of the decaying transients (or the settling time) is long compared to the interaction time Δt, the system is not yet ready for rigid-body approximations until a time long after the contact period. We might be tempted to consider the system to be rigid in an *average* sense, in that it has well-defined *mean* rigid-body motions, before and after the application of the impulse, although the vibrations persist for a long time. However, for general two and three dimensional bodies that are, say, vibrating for a long time, it may not be possible to define a 'mean' equivalent rigid-body motion.

For example, consider two equal point masses joined by a spring of nonzero free length, moving in a plane. Let initial conditions be such that the center of mass is stationary, while the angular momentum of the system about its center of mass is H (a conserved quantity). Each point mass executes central force motion. Let the radial position of each point mass oscillate about its mean value with a small amplitude A. In that case, the *average* angular velocity of the system can be shown to be of the form $C_0 + C_1 A^2$, for suitable constants C_0 and C_1. The small correction to the mean angular velocity produces large changes in orientation over long enough times. Thus, if two bodies with identical mass distribution are given identical impulses, and if the vibrations in them die out over widely different times, then eventually their orientations may be quite different. If the amplitude of vibrations is large (say, comparable to the size of the body), then the orientations of the vibrating and non-vibrating object may differ appreciably after a short time.

The previous discussion shows some of the angular momentum-related difficulties in 2D and 3D which do not appear in 1D, if vibrations have large amplitudes and/or last long. Note that the position and velocity of the center of mass do not suffer such ambiguities: unlike angular momentum, linear momentum is the derivative of a function of the system's coordinates.

There is another way in which persistent vibrations can violate a rigid body modeling approach (J. Jenkins — private communication). If we want only a smaller (finite or $\mathcal{O}(1)$) time prediction, then small persistent vibrations will not lead to large errors. However, if the body has another collision before the vibrations are dead, then some vibrational kinetic energy may get transferred back to rigid body modes. Thus, a model for the *second* collision will need to account for the vibrations of the object. In a general rigid body dynamics simulation environment, vibrations need to die out on a time scale that is short compared to the typical time between collisions, i.e., the time scale of overall motions.

Thus, due to considerations of angular momentum for long-time predictions, and energy for short-time predictions where several collisions might occur sequentially, impulse-response rigidity is the weakest sense in which an object may be considered as rigid in general 2D and 3D collisions.

9 Algebraic, incremental and full-body deformation rigid body collision laws

There are three general approaches to finding the net resultant collisional impulse.

In *incremental* collision laws, rigid body force-acceleration and angular momentum balance equations are used to describe the dynamics *during* the collision. That is, incremental collision laws use force-response rigidity. The collision is modeled using a pseudo-static local-deformation calculation interacting with a mass matrix $\mathbf{M}$ (or a rigid body dynamics calculation equivalent to using $\mathbf{M}$).

Incremental laws can roughly be divided into two classes. In the first the contact region is described by a simplistic combination of simple elements such as springs and dashpots. In the second class the contact region is modeled using continuum mechanics. This class includes, for example, models based on Hertz contact with or without distributed friction.

Incremental rigid body collision laws, such as Routh's law (Routh, 1897), Hertz-contact based treatments (e.g., Maw *et al.,* 1981; Jaeger, 1994), as well as more *ad hoc* spring-dashpot type approaches (e.g., Goyal *et al.*, 1994) *all* depend on force-response rigidity, whether such an assumption is justified or not. Even when force-response rigidity may be an inaccurate description, these laws might still be used to calculate a plausible collisional impulse. That is, force-response based predictions using standard passive contact elements will respect basic constraints like nonnegative energy dissipation, non-interpenetration and (if applicable) the Coulomb friction inequality at the impulse level. However, if the incremental model is not a good approximation of the actual contact mechanics, and/or if the body is not well approximated as force-response rigid, incremental laws should not be expected to make accurate predictions.

In *algebraic* laws, the impulse-momentum equations (Eq. 2 or equivalent rigid body calculations) are used to summarize the mechanics of the bodies. The *net* interaction is predicted without paying attention to the detailed dynamic interaction during the collision. Algebraic laws are generalizations of the familiar 1D Newton or Poisson laws based on a coefficient of restitution e. Algebraic rigid body collision laws (such as Whittaker's (1944) or Kane and Levinson's (1985) collision law, Smith's (1991) law, the laws proposed in Brach's text (1991), or the new law proposed in Chatterjee and Ruina (1998)), assume impulse-response rigidity. Note that algebraic collision laws do not deny the existence of deformations during the collisions. They just do not pay attention to their details. As is commonly acknowledged (see, e.g., Smith 1991), any given algebraic rigid body collision law should be looked upon as an approximate treatment[5].

In *full-body-deformation* collision treatments the deformations of the full bodies are modeled during the collision. If it is assumed that at times before and soon after the collision the overall motions of the bodies have settled to something well described by rigid-body kinematics then these calculations may still be viewed as a means to finding the collisional impulse between two nominally rigid bodies. The well-known one dimensional wave propagation model of longitudinally colliding elastic bars falls in this class. See also the model used by Stoianovici and Hurmuzlu (1996).

The algebraic and incremental approaches to calculating collisional impulses are sometimes called, respectively, 'hard' and 'soft' (Goyal *et al.*, 1994; Walton, 1992). Algebraic laws are called 'hard' because they pay no attention to the details of even the contact-region deformation. Incremental laws are called 'soft' because they follow the details of some contact compliance model. The words 'soft' and 'hard' are somewhat inverted in meaning, however. 'Soft' collision laws actually depend on the stronger force-response rigidity (the whole body, but for a small contact region, behaves like a rigid body throughout the collision) while the 'hard' collision laws depend only on the weaker impulse-response rigidity (rigid motions only before and soon after the collision).

The following examples may further clarify some of these collisional rigidity issues.

9.1 Routh's Law

Routh's law has received attention from many authors (e.g., Mayer, 1902; Plyavniyeks, 1970; Keller, 1986; Wang and Mason, 1992; Bhatt and Koechling, 1995; Batlle, 1996). In addition to force-response rigidity, this law assumes that the contact interaction is well-described by a (nonzero) finite compliance in the normal direction along with *zero* compliance in the tangential direction. Errors in predictions from Routh's model can therefore arise from tangential compliance effects or other contact model defects, even for objects accurately modeled as force-response rigid.

[5]This is not to say that special-purpose algebraic collision laws for special objects in particular configurations cannot be accurate (see, e.g., a special law for collisions between slender rods and massive objects presented in Hurmuzlu, 1997).

9.2 Robots and Linkages

One might assume that colliding robots/linkages are rigid with ideal geometric constraints (see e.g., Bhatt and Koechling, 1993; Marghitu and Hurmuzlu, 1995; Mills and Nguyen, 1992). Under such an assumption, incremental rigid body collision laws might be used. However, real machines are frequently made of long, slender components which are not expected to be force-response rigid. Moreover, small imperfections like finite (nonzero) bearing clearances complicate matters further. To be effectively force-response rigid in a collision, a mechanism must have suitably rigid components *and* the bearing clearances and compliance effects have to be negligible compared to the relative displacements in the contact region. Real mechanisms in colliding contact will typically *not* be well approximated as force-response rigid. However, with low joint friction, impulse-response rigidity may be quite accurate (for these objects the tensor $\mathbf{M}$ is still second order, or 3×3, but depends on all the rigid bodies in the chain).

9.3 Examples of nonlocal deformations in 'rigid body' collisions

Example 1 (Elastic vibrations): Global deformations associated with elastic vibration are expected to be important for bodies when the net contact interaction time is not large compared to the lowest natural vibration frequencies. Short contact times can come from conforming contact in shock loading or when a body is hit with a much smaller or stiffer body. On the other hand, when an object is much larger in one dimension than in others, as for rods, relatively large-time vibrational modes are expected. Accurate models might still be obtained by considering the small, localized interaction in the contact region to be pseudostatic (i.e., using mechanisms like nonlinear springs, dashpots, and friction elements, all without inertia), as demonstrated for the case of slender rods colliding with a massive anvil by Stoianovici and Hurmuzlu (1996). They found that the coefficient of restitution depends strongly on the orientation of the rods because of the importance of bending vibrations and consequent multiple micro impacts for each macroscopic collision. Such behavior was not foreseen in several papers on general collisions where slender rods are used as 'example' problems (Brach, 1989; Stronge, 1990; Wang and Mason, 1992). The treatments in these papers are not wrong for 'rigid' rods in that they do not violate laws or principles of mechanics. The treatments are just inaccurate for these slender steel rods. One might think (Hurmuzlu, 1996) that such collisions should therefore not be treated as rigid body collisions. However, several essential features of rigid body mechanics are retained (Chatterjee and Ruina, 1997). Rather the specific models considered (Brach, 1989; Stronge, 1990; Wang and Mason, 1992) are implicitly and inappropriately based on force-response rigidity. However models based on impulse-response rigidity (Hurmuzlu, 1997) or on full deformations (Stoianovici and Hurmuzlu, 1996) can still be usefully applied to determine the relation between the before-collision and after-collision rigid motions.

Example 2 (Disks): In Chatterjee (1997) and Calsamiglia *et al.* (1997), results are given of collision experiments with thin disks against a massive steel plate. Practically all rigid body collision models, for the special case of sliding collisions in 2D, predict that the tangential component of impulse transmitted is equal to μ (coefficient of friction) times the normal component. This includes, for example, the models in Routh (1897), Whittaker (1944), or more recent models as in Smith (1991)[6]. This idea is generally accepted as correct, particularly for spheres and disks, and has been verified for some cases (Maw *et al.*, 1981). However, the new data (Chatterjee, 1997; Calsamiglia *et al.,* 1997) indicates that for thin enough disks, *even for* collisions where the tangential motion is not reversed or stopped (i.e., in the 'sliding range'), the ratio of tangential to normal impulses is not constant – it can vary by roughly a factor of two. Only in the limiting case of grazing incidence, for the disks studied, does the impulse

[6]But not including Brach's approach (1991) where the ratio of impulses is not predicted but assumed to be an input parameter known *a priori.*

ratio become approximately equal to the friction coefficient measured in separate experiments. The discrepancy between the standard frictional prediction and the experimental result may stem in part from the importance of the whole-body deformation during collision (see discussion in Chatterjee (1997) or Calsamiglia *et al.* (1997)). We remark that for the disks, normal restitution was roughly constant for all incidence angles, as usually expected, and in contrast to the slender rods mentioned earlier.

Again, the thin disk experiments do not negate the concept of a rigid body collision law, but point out shortcomings in particular laws and the need for care in the use of the term 'rigidity' when only impulse-response rigidity may apply.

Example 3 (Pseudo-rigid bodies): A special theoretical approach to collisions uses pseudo-rigid bodies (e.g., Cohen and Mac Sithigh, 1991), where the strain in a colliding object is uniform (i.e., the strain is a function of time but not position). Pseudo-rigid bodies are a special case of using a finite number of approximate modes to represent the global deformations in the objects. This approach does not allow the possibilities of dominant localized deformations (as in, e.g., collisions of spheres or other roundish, solid objects); of dominant stress waves (as in, e.g., the longitudinal collisions of long rods); and of slow bending vibrations (as in, e.g., the transverse collisions of slender rods, plates, etc.). In other words, the choice of mode shapes used in the pseudo-rigid body approach may not be appropriate for many collisions; however, as in the case of incremental models, this approach provides a means to calculate a plausible collision impulse that will satisfy basic restrictions, whether or not the object being modeled is well approximated as pseudo-rigid. A pseudo-rigid body is not force-response rigid but might be impulse-response rigid if the vibrations are assumed to damp out soon after collision.

10 Summary

The prediction of a collision model for nominally rigid bodies depends on the momentum balance equations and constitutive laws. Depending on the degree to which rigid body kinematics is followed by the real body, different types of constitutive laws are appropriate. The degree to which rigid body kinematics is followed can be summarized by the extent to which the Newton-Euler equations for a rigid body describe the motion of a near-contact reference point in response to the collision contact force. If the Newton-Euler equations, exactly equivalent for collisions to Eq. 1, apply through the collision, then the body is *force-response* rigid. If rigid body dynamics equations only accurately describe the net changes in velocity, as expressed by Eq. 2, then the bodies are *impulse-response* rigid. Force-response bodies are impulse-response rigid, but impulse-response bodies may not be force-response rigid. Collision laws that use a micro-mechanical contact model interacting with rigid bodies assume force-response rigidity, whether the assumption is justified or not. Algebraic laws, or collisions calculated using bulk deformation, depend only on impulse-response rigidity.

If incremental collision laws are applied when force-response rigidity is not accurate, or if they use an inaccurate micro-mechanical model, then they can predict plausible collisional outcomes but should not be expected to be more or less accurate than algebraic collision laws.

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List of Figures

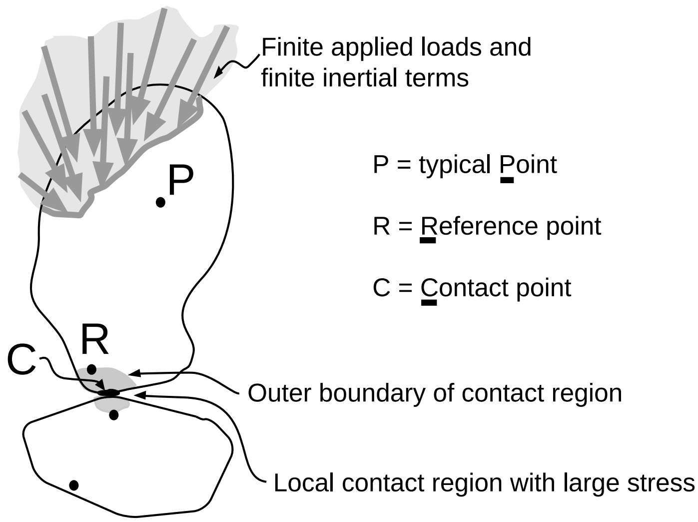

Figure 1: Schematic diagram of generic 3D collision. At issue is to what extent the points C, R, and P on one body may be regarded as rigidly linked.

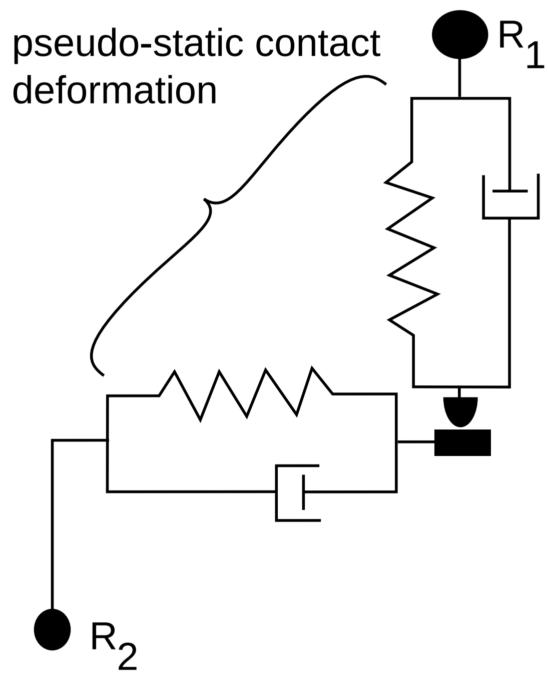

Figure 2: Schematic diagram showing the deformations in the contact region. Contact deformation region acts as a massless, pseudo-static interaction mechanism between the reference points R_1 and R_2 on the two bodies. The interaction mechanism can be any passive mechanism including e.g., spring/dashpot/friction elements or Hertz contact.

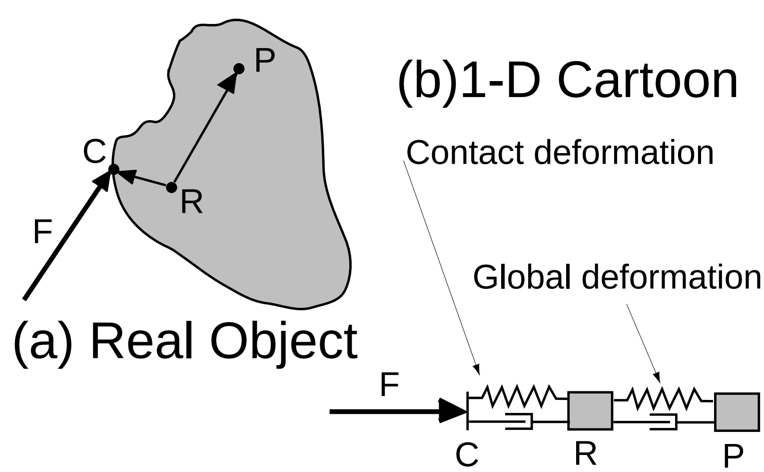

Figure 3: (a) Force F acts at point C on an object – R is a reference point, and P is a typical point; (b) 1D Cartoon: between C and R is contact deformation and between R and P is deformation of the full body.

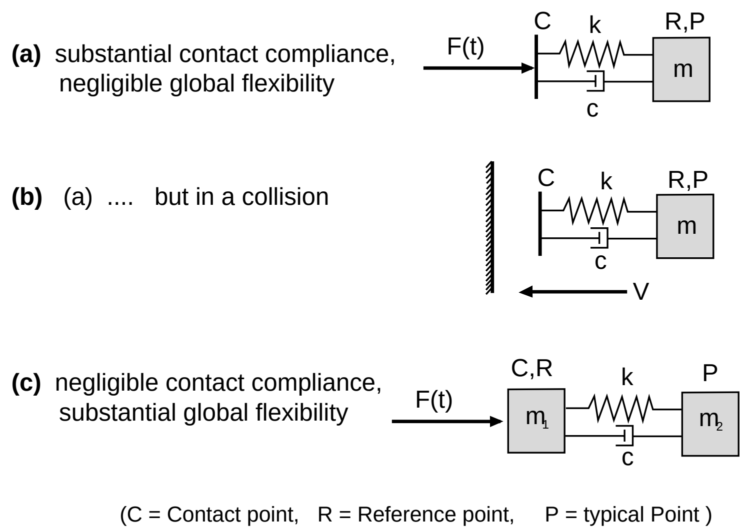

Figure 4: (a) 1-D model with negligible global flexibility so that R and P are modeled as on the same rigid body but there is substantial contact compliance; (b) same model as '(a)' but in a collisional context; (c) 1-D model with negligible contact compliance and substantial global flexibility.

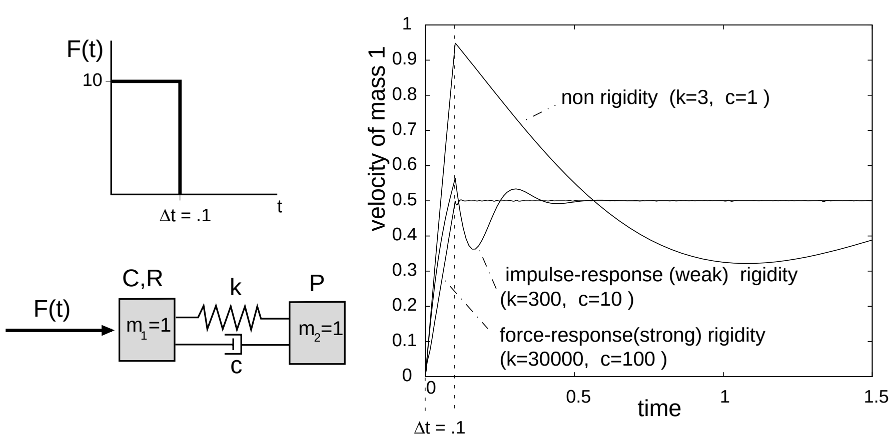

Figure 5: Simulation of two mass system: $F = 10, \Delta t = 0.1, P = 1, m_1 = m_2 = 1$ and k, C as shown in figure.