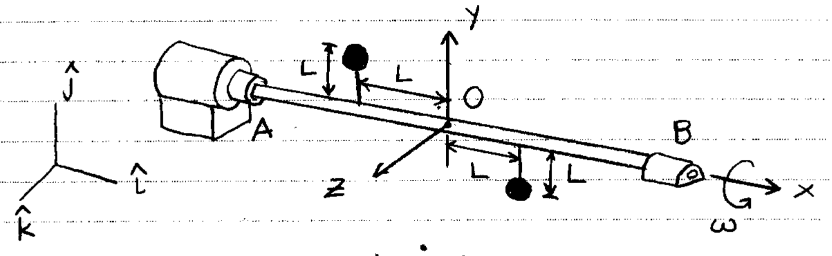
TAM 203 - Spring, 2000
HW #9 Solutions

8.23 The two mass system shown has angular velocity $\underline{\omega} = \omega\hat{\imath}$. At the instant shown, masses are in the xy-plane.



a) What are $\underline{L}$ & $\dot{\underline{L}}$?

$$x_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{m(-L)+m(L)}{m+m} = 0$$

$$y_{CM} = \frac{\sum m_i y_i}{\sum m_i} = \frac{m(L)+m(-L)}{m+m} = 0$$

$\Rightarrow$ CM on the axis of rotation

$\Rightarrow$ $\underline{v}_{CM} = \underline{0}$, $\underline{a}_{CM} = \underline{0}$

$$\therefore \boxed{\dot{\underline{L}} = \underline{0}, \quad \underline{L} = \underline{0}}$$

b) What are $\underline{H}_{/O}$ & $\dot{\underline{H}}_{/O}$?

$$\underline{H}_{/O} = \sum_{i=1}^{2} \underline{r}_{i/O} \times m_i \underline{v}_i \quad \text{where } \underline{v}_i = \underline{\omega} \times \underline{r}_{i/O}$$

$$= \underline{r}_{1/O} \times m(\underline{\omega} \times \underline{r}_{1/O}) + \underline{r}_{2/O} \times m(\underline{\omega} \times \underline{r}_{2/O})$$

$$= L(-\hat{\imath}+\hat{\jmath}) \times m[\omega\hat{\imath} \times L(-\hat{\imath}+\hat{\jmath})]$$

$$+ L(\hat{\imath}-\hat{\jmath}) \times m[\omega\hat{\imath} \times L(\hat{\imath}-\hat{\jmath})]$$

$$= L(-\hat{\imath}+\hat{\jmath}) \times m\omega L\hat{k} + L(\hat{\imath}-\hat{\jmath}) \times -m\omega L\hat{k}$$

$$= m\omega L^2(\hat{\jmath}+\hat{\imath}) + m\omega L^2(\hat{\jmath}+\hat{\imath})$$

$$\therefore \boxed{\underline{H}_{/O} = 2m\omega L^2(\hat{\imath}+\hat{\jmath})}$$

$$\dot{\underline{H}}_{/O} = \sum_{i=1}^{2} \underline{r}_{i/O} \times m_i \underline{a}_i \quad \text{where } \underline{a}_i = \underline{\omega} \times (\underline{\omega} \times \underline{r}_{i/O})$$

$$= \underline{\omega} \times \underline{v}_i$$

$$= \underline{r}_{1/O} \times m(\underline{\omega} \times \underline{v}_1) + \underline{r}_{2/O} \times m(\underline{\omega} \times \underline{v}_2)$$

$$= L(-\hat{\imath}+\hat{\jmath}) \times m[\omega\hat{\imath} \times m\omega L\hat{k}]$$

$$+ L(\hat{\imath}-\hat{\jmath}) \times m[\omega\hat{\imath} \times -m\omega L\hat{k}]$$

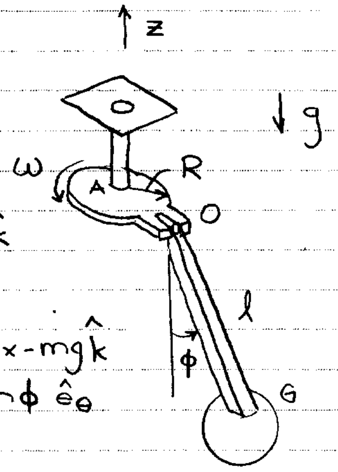
$$= L(-\hat{\imath}+\hat{\jmath}) \times -m\omega^2 L\hat{\jmath} + L(\hat{\imath}-\hat{\jmath}) \times m\omega^2 L\hat{\jmath}$$

$$= m\omega^2 L^2\hat{k} + m\omega^2 L^2\hat{k}$$

$$\therefore \boxed{\dot{\underline{H}}_{/O} = 2m\omega^2 L^2\hat{k}}$$

A turntable rotates @ constant rate, ω, about the z-axis. At the edge swings a pendulum which makes a constant angle ϕ w/ the z-axis. The pendulum consists of a pt. mass connected to a massless rod.

a) Find an equation for ϕ.



$\underline{AMB/_o}$: $\Sigma \underline{M}_{/o} = \dot{\underline{H}}_{/o}$

$$\Sigma \underline{M}_{/o} = \underline{M} + \underline{r}_{G/o} \times -mg\hat{k}$$
$$= M_r \hat{e}_R + M_z \hat{k}$$
$$+ \ell(\sin\phi \hat{e}_R - \cos\phi \hat{k}) \times -mg\hat{k}$$
$$= M_r \hat{e}_R + M_z \hat{k} + mg\ell \sin\phi \, \hat{e}_\theta$$

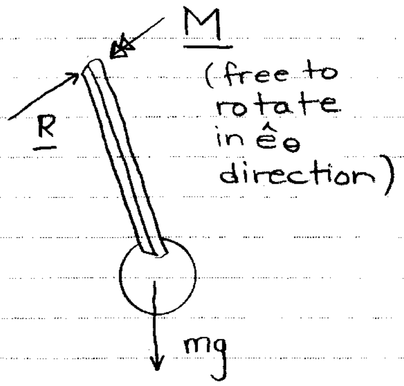
$$\dot{\underline{H}}_{/o} = \underline{r}_{G/o} \times m\underline{a}$$

where $\underline{a}_G = \underline{\omega} \times (\underline{\omega} \times \underline{r}_{G/A})$

$$\Rightarrow \underline{a}_G = \omega\hat{k} \times [\omega\hat{k} \times (r\hat{e}_R - \ell\cos\phi \, \hat{k})]$$
$$= \omega\hat{k} \times \omega r \hat{e}_\theta$$
$$= -\omega^2 r \hat{e}_R$$

$$\Rightarrow \dot{\underline{H}}_{/o} = \ell(\sin\phi \hat{e}_R - \cos\phi \hat{k}) \times -m\omega^2 r \hat{e}_R$$
$$= m\omega^2 \ell r \cos\phi \, \hat{e}_\theta$$

FBD



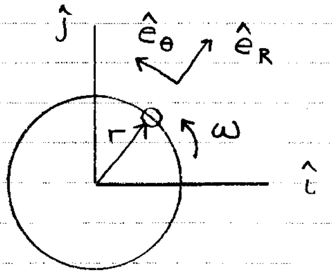
$AMB/_o \cdot \hat{e}_\theta$

$$\Rightarrow \cancel{m}g\cancel{\ell}\sin\phi = \cancel{m}\omega^2\cancel{\ell} r\cos\phi$$

$$g\sin\phi = \omega^2 r\cos\phi$$
$$= \omega^2(R + \ell\sin\phi)\cos\phi$$

$$\therefore \boxed{\tan\phi = \frac{\omega^2}{g}(R + \ell\sin\phi)}$$

Top view of pt. mass trajectory



where $\underline{r = R + \ell\sin\phi}$

b) Find all possible angles for steady circular motion.

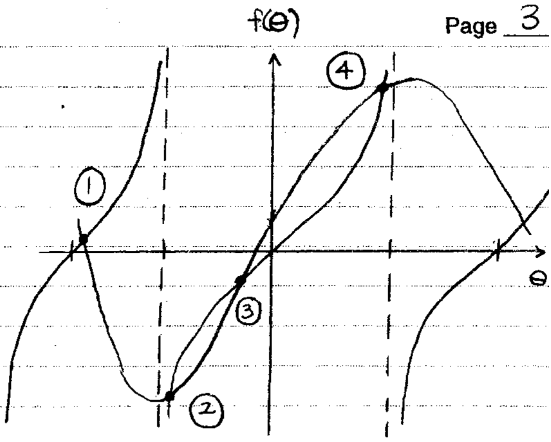
$$\begin{cases} \omega = 2 \text{ rad/s}, \ R = 2\text{ m} \\ g = 10 \text{ m/s}^2, \ \ell = 10\text{ m} \end{cases}$$

get $\tan\phi = \frac{4}{5}(1 + 5\sin\phi)$

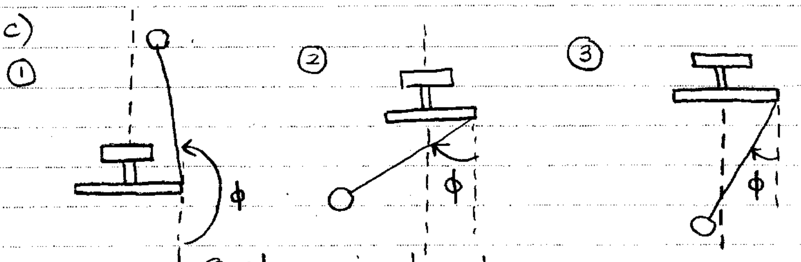
(continued)

8.26 contd.

Plots of $\tan\phi$ and $\frac{4}{5}(1+5\sin\phi)$ show 4 intersections $\Rightarrow$ 4 possible angles for steady motion.

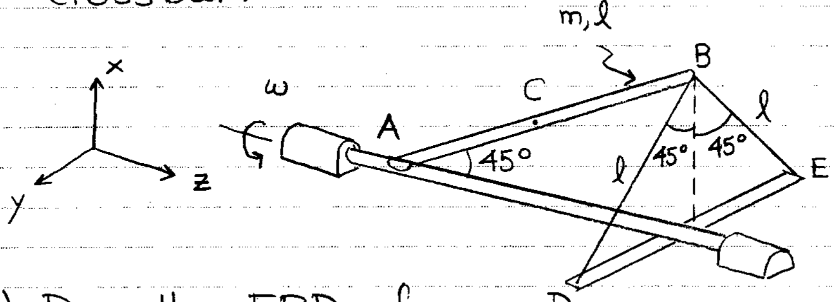


c)



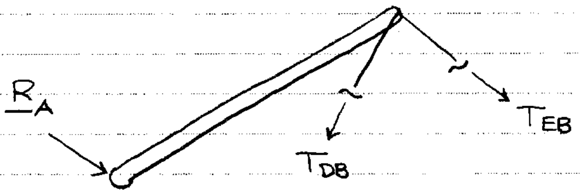
④ shown in book

8.44 A rod is attached to a shaft which spins @ constant rate ω about the z-axis. There is a ball and socket joint @ A and B is held by 2 strings connected to a crossbar.



a) Draw the FBD of the rod.

FBD



b) Find T_{DB}.

i) LMB: $\Sigma \underline{F} = m\underline{a}_c$

where
$$\underline{a}_c = \underline{\omega} \times (\underline{\omega} \times \underline{r}_{c/A})$$
$$= \omega\hat{k} \times [\omega\hat{k} \times \tfrac{l}{2\sqrt{2}}(\hat{\imath}+\hat{k})]$$
$$= \omega\hat{k} \times \tfrac{\omega l}{2\sqrt{2}}\hat{\jmath}$$

(continued)

$$\Rightarrow \underline{a}_C = -\frac{\omega^2 \ell}{2\sqrt{2}}\,\hat{\imath}$$

$$\Sigma\underline{F} = \underline{R}_A + T_{DB}\,\hat{\lambda}_{DB} + T_{EB}\,\hat{\lambda}_{EB}$$

where $\hat{\lambda}_{DB} = \dfrac{\underline{r}_{D/B}}{|\underline{r}_{D/B}|} = \dfrac{\ell\hat{\jmath} - \ell\hat{\imath}}{\ell\sqrt{2}} = \dfrac{1}{\sqrt{2}}(\hat{\jmath} - \hat{\imath})$

$\hat{\lambda}_{EB} = \dfrac{\underline{r}_{E/B}}{|\underline{r}_{E/B}|} = \dfrac{-\ell\hat{\jmath} - \ell\hat{\imath}}{\ell\sqrt{2}} = \dfrac{1}{\sqrt{2}}(-\hat{\jmath} - \hat{\imath})$

• Using LMB, we get 3 equations for 5 unknowns ($\underline{R}_A$, T_{DB}, T_{EB}). So, take AMB about, say, pt. A:

$$\text{AMB}_{/A}: \quad \Sigma\underline{M}_{/A} = \dot{\underline{H}}_{/A}$$

$$\Sigma\underline{M}_{/A} = \underline{r}_{B/A} \times T_{DB}\,\hat{\lambda}_{DB} + \underline{r}_{B/A} \times T_{EB}\,\hat{\lambda}_{EB}$$

$$= \frac{\ell}{\sqrt{2}}(\hat{k} + \hat{\imath}) \times \frac{1}{\sqrt{2}} T_{DB}(\hat{\jmath} - \hat{\imath}) + \frac{\ell}{\sqrt{2}}(\hat{k} + \hat{\imath}) \times \frac{1}{\sqrt{2}} T_{EB}(-\hat{\jmath} - \hat{\imath})$$

$$= \frac{1}{2} T_{DB}\,\ell\,(-\hat{\imath} - \hat{\jmath} + \hat{k}) + \frac{1}{2} T_{EB}\,\ell\,(\hat{\imath} - \hat{\jmath} - \hat{k})$$

$$\dot{\underline{H}}_{/A} = \int \underline{r}_{/A} \times \underline{a}\; dm$$

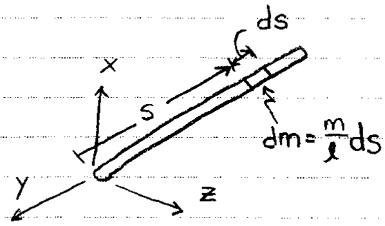
$$= \int \left[\frac{1}{\sqrt{2}} s(\hat{k} + \hat{\imath}) \times \underbrace{-\frac{1}{\sqrt{2}} s\omega^2\,\hat{\imath}}_{\text{similar to } \underline{a}_C \text{ above w/ } \frac{\ell}{2} \mapsto s}\right] \frac{m}{\ell}\, ds$$

$$= \left(\int_0^{\ell} -\frac{1}{2}\omega^2 \frac{m}{\ell} s^2\, ds\right)\hat{\jmath}$$

$$= -\frac{m\omega^2}{\ell}\int_0^{\ell} s^2\, ds\;\hat{\jmath}$$

$$= -\frac{m\omega^2}{\ell}\left(\frac{1}{3}\ell^3\right)\hat{\jmath}$$

$$= -\frac{1}{6} m\omega^2 \ell^2\,\hat{\jmath}$$



$\text{AMB}_{/A} \cdot \hat{\imath} \Rightarrow -\frac{1}{2} T_{DB}\,\ell + \frac{1}{2} T_{EB}\,\ell = 0 \quad (1)$
(& $\text{AMB}_{/A} \cdot \hat{k}$)

$\text{AMB}_{/A} \cdot \hat{\jmath} \Rightarrow -\frac{1}{2} T_{DB}\,\ell - \frac{1}{2} T_{DB}\,\ell = -\frac{1}{6} m\omega^2 \ell^2$

(1) $\Rightarrow T_{DB} = T_{EB}$

(2) $\Rightarrow -T_{DB}\,\ell = -\frac{1}{6} m\omega^2 \ell$

$$\therefore \boxed{T_{DB} = \frac{1}{6} m\omega^2 \ell}$$

(continued)

ii) $\underline{\text{AMB}}$/axis AE : $\{\Sigma \underline{M}_{/A} = \dot{\underline{H}}_{/A}\} \cdot \hat{\lambda}_{EA}$

where $\hat{\lambda}_{EA} = \frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$

$$\Sigma \underline{M}_{/A} \cdot \hat{\lambda}_{EA} = \tfrac{1}{2} T_{DB} \frac{l}{\sqrt{2}}(1+1) + \tfrac{1}{2} T_{EB} \frac{l}{\sqrt{2}}(1-1)$$
$$= T_{DB}\frac{l}{\sqrt{2}}$$

$$\dot{\underline{H}}_{/A} \cdot \hat{\lambda}_{EA} = \frac{1}{\sqrt{2}} \frac{1}{6} m\omega^2 l$$

$$\Rightarrow \frac{l}{\sqrt{2}} T_{DB} = \frac{1}{\sqrt{2}} \frac{1}{6} m\omega^2 l$$

$$\therefore \boxed{T_{DB} = \tfrac{1}{6} m\omega^2 l}$$

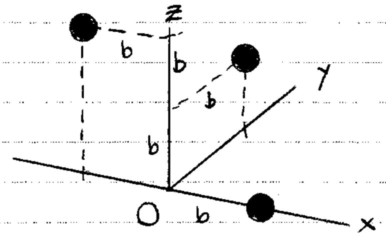
• same as above using just one equation

c) Find $\underline{R}_A$ by using LMB in b) i) :

$$\underline{R}_A = -T_{DB}\hat{\lambda}_{DB} - T_{EB}\hat{\lambda}_{EB} + m\underline{a}_c$$
$$= -\tfrac{1}{6} m\omega^2 l \frac{1}{\sqrt{2}}(\hat{j}-\hat{i}) - \underbrace{\tfrac{1}{6} m\omega^2 l \frac{1}{\sqrt{2}}(-\hat{j}-\hat{i})}_{\text{from (1)}}$$
$$- \frac{m}{2\sqrt{2}}\omega^2 l \,\hat{i}$$

$$\Rightarrow \boxed{\underline{R}_A = -\frac{\sqrt{2}}{12} m\omega^2 l \,\hat{i}}$$

8.49 Three identical solid spheres are located as shown. Find $[I^o]$.



Strategy:
Find $[I^o]$ for each sphere using ∥-axis thm. and add them up.

$$[I^o] = [I^{CM}] + m\begin{bmatrix} y_{CM/o}^2 + z_{CM/o}^2 & -x_{CM/o}y_{CM/o} & -x_{CM/o}z_{CM/o} \\ -x_{CM/o}y_{CM/o} & x_{CM/o}^2 + z_{CM/o}^2 & -y_{CM/o}z_{CM/o} \\ -x_{CM/o}z_{CM/o} & -y_{CM/o}z_{CM/o} & x_{CM/o}^2 + y_{CM/o}^2 \end{bmatrix}$$

① $$[I^o]_1 = \begin{bmatrix} \frac{2}{5}mR^2 & 0 & 0 \\ 0 & \frac{2}{5}mR^2 & 0 \\ 0 & 0 & \frac{2}{5}mR^2 \end{bmatrix}$$

$$+m\begin{bmatrix} 0^2+(2b)^2 & -(-b)(0) & -(-b)(2b) \\ -(-b)(0) & b^2+(2b)^2 & -(0)(2b) \\ -(-b)(2b) & -(0)(2b) & b^2+0^2 \end{bmatrix}$$

(continued)

$$\Rightarrow [I^o]_1 = m \begin{bmatrix} \frac{2}{5}R^2+4b^2 & 0 & 2b^2 \\ 0 & \frac{2}{5}R^2+5b^2 & 0 \\ 2b^2 & 0 & \frac{2}{5}R^2+b^2 \end{bmatrix}$$

② $$[I^o]_2 = \begin{bmatrix} \frac{2}{5}mR^2 & 0 & 0 \\ 0 & \frac{2}{5}mR^2 & 0 \\ 0 & 0 & \frac{2}{5}mR^2 \end{bmatrix}$$

$$+m \begin{bmatrix} 0^2+0^2 & -(b)(0) & -(b)(0) \\ -(b)(0) & b^2+0^2 & -(0)(0) \\ -(b)(0) & -(0)(0) & b^2+0^2 \end{bmatrix}$$

$$\Rightarrow [I^o]_2 = m \begin{bmatrix} \frac{2}{5}R^2 & 0 & 0 \\ 0 & \frac{2}{5}R^2+b^2 & 0 \\ 0 & 0 & \frac{2}{5}R^2+b^2 \end{bmatrix}$$

③ $$[I^o]_3 = \begin{bmatrix} \frac{2}{5}mR^2 & 0 & 0 \\ 0 & \frac{2}{5}mR^2 & 0 \\ 0 & 0 & \frac{2}{5}mR^2 \end{bmatrix}$$

$$+m \begin{bmatrix} b^2+b^2 & -(0)(b) & -(0)(b) \\ -(0)(b) & 0^2+b^2 & -(b)(b) \\ -(0)(b) & -(b)(b) & 0^2+b^2 \end{bmatrix}$$

$$\Rightarrow [I^o]_3 = m \begin{bmatrix} \frac{2}{5}R^2+2b^2 & 0 & 0 \\ 0 & \frac{2}{5}R^2+b^2 & -b^2 \\ 0 & -b^2 & \frac{2}{5}R^2+b^2 \end{bmatrix}$$

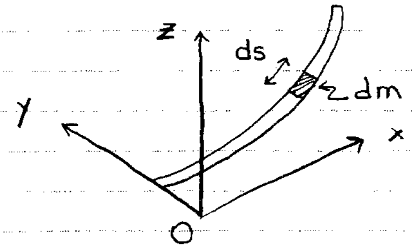
$$\therefore \quad [I^o] = [I^o]_1 + [I^o]_2 + [I^o]_3$$

$$\boxed{[I^o] = m \begin{bmatrix} \frac{6}{5}R^2+6b^2 & 0 & 2b^2 \\ 0 & \frac{6}{5}R^2+7b^2 & -b^2 \\ 2b^2 & -b^2 & \frac{6}{5}R^2+3b^2 \end{bmatrix}}$$

Problem 5

A curved rod of mass $m = 3\,kg$ is along the curve $x = \sin t$, $y = e^t$, $z = t^2$ for $0 \le t \le 2$. What is $[I^o]$?

Look at a small segment of the rod:



$$dm = \rho\, ds = \frac{m}{\ell}\, ds$$

where $\ell = \int_{t=0}^{t=2} ds$ and $ds^2 = dx^2 + dy^2 + dz^2$

(continued)

Note that ds can also be written as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \sqrt{\cos^2 t + e^{2t} + 4t^2}\, dt$$

$$\therefore \ l = \int_0^2 \sqrt{\cos^2 t + e^{2t} + 4t^2}\, dt$$

Now get elements of $[I^o]$:

$$I_{xx}^o = \int (y_{/0}^2 + z_{/0}^2)\, dm$$
$$= \rho \int_0^2 (e^{2t} + t^4) \sqrt{\cos^2 t + e^{2t} + 4t^2}\, dt$$

$$I_{yy}^o = \int (x_{/0}^2 + z_{/0}^2)\, dm$$
$$= \rho \int_0^2 (\sin^2 t + t^4) \sqrt{\qquad\qquad}\, dt$$

$$I_{zz}^o = \int (x_{/0}^2 + y_{/0}^2)\, dm$$
$$= \rho \int_0^2 (\sin^2 t + e^{2t}) \sqrt{\qquad\qquad}\, dt$$

$$I_{xy}^o = -\int x_{/0}\, y_{/0}\, dm$$
$$= -\rho \int_0^2 (\sin t) e^t \sqrt{\qquad\qquad}\, dt$$

$$I_{xz}^o = -\int x_{/0}\, z_{/0}\, dm$$
$$= -\rho \int_0^2 (\sin t) t^2 \sqrt{\qquad\qquad}\, dt$$

$$I_{yz}^o = -\int y_{/0}\, z_{/0}\, dm$$
$$= -\rho \int_0^2 t^2 e^t \sqrt{\qquad\qquad}\, dt$$

To calculate these integrals, you can write your own integration routine (e.g. trapezoidal rule, Simpson's rule, etc.) or use Matlab's QUAD or QUAD8 functions.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This script file determines the moment of inertia
% matrix for the final problem in HW#9.  Calls are
% made to Matlab's quad8 function which does
% numerical integration of the given function 'func
% from x1 to x2.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

m=3;   % mass of rod in kg

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% To use 'quad8' you must supply a function, i.e.
% the integrand, which must return a vector.  You
% must also give the limits of integration (here we
% integrate from t=0 to t=2).
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

L=quad8('arclength',0,2);  % length of the rod (m)
rho=m/L;                   % rod density (kg/m)

Ixx=rho*quad8('Ixx_func',0,2);
Ixy=rho*quad8('Ixy_func',0,2);
Ixz=rho*quad8('Ixz_func',0,2);
Iyy=rho*quad8('Iyy_func',0,2);
Iyz=rho*quad8('Iyz_func',0,2);
Izz=rho*quad8('Izz_func',0,2);

I=[Ixx Ixy Ixz;
   Ixy Iyy Iyz;
   Ixz Iyz Izz]

% Answer:
% [I]=[ 76.8631  -11.2810   -5.5291
%      -11.2810   18.1371  -30.8026
%       -5.5291  -30.8026   63.2581 ]  kg m^2

% The following are separate function files:

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=arclength(t)
f=sqrt(cos(t).^2+exp(2*t)+4*t.^2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Ixx_func(t)
f=(exp(2*t)+t.^4).*sqrt(cos(t).^2+exp(2*t)+4*t.^2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Ixy_func(t)
f=-sin(t).*exp(t).*sqrt(cos(t).^2+exp(2*t)+4*t.^2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Ixz_func(t)
f=-sin(t).*t.^2.*sqrt(cos(t).^2+exp(2*t)+4*t.^2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Iyy_func(t)
f=(sin(t).^2+t.^4).*sqrt(cos(t).^2+exp(2*t)+4*t.^2)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Iyz_func(t)
f=-exp(t).*t.^2.*sqrt(cos(t).^2+exp(2*t)+4*t.^2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Izz_func(t)
f=(exp(2*t)+sin(t).^2).*sqrt(cos(t).^2+exp(2*t)+
   4*t.^2);
```