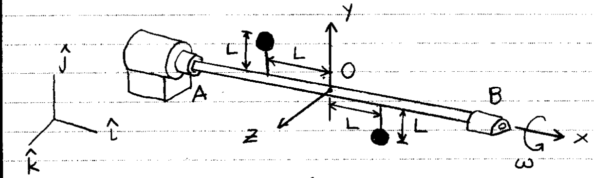
TAM 203 - Spring, 2000
HW #9 Solutions

8.23 The two mass system shown has angular velocity $\underline{\omega} = \omega\hat{i}$. At the instant shown, masses are in the xy-plane.



a) What are $\underline{L}$ & $\dot{\underline{L}}$?

$$x_{CM} = \frac{\Sigma m_i x_i}{\Sigma m_i} = \frac{m(-L)+m(L)}{m+m} = 0$$

$$y_{CM} = \frac{\Sigma m_i y_i}{\Sigma m_i} = \frac{m(L)+m(-L)}{m+m} = 0$$

$\Rightarrow$ CM on the axis of rotation

$\Rightarrow$ $\underline{v}_{CM} = \underline{0}$, $\underline{a}_{CM} = \underline{0}$

$$\therefore \boxed{\dot{\underline{L}} = \underline{0},\ \underline{L} = \underline{0}}$$

b) What are $\underline{H}_{/o}$ & $\dot{\underline{H}}_{/o}$?

$\underline{H}_{/o} = \sum_{i=1}^{2} \underline{r}_{i/o} \times m_i \underline{v}_i$ where $\underline{v}_i = \underline{\omega} \times \underline{r}_{i/o}$

$$= \underline{r}_{1/o} \times m(\underline{\omega}\times\underline{r}_{1/o}) + \underline{r}_{2/o}\times m(\underline{\omega}\times\underline{r}_{2/o})$$
$$= L(-\hat{i}+\hat{j})\times m[\omega\hat{i}\times L(-\hat{i}+\hat{j})]$$
$$+ L(\hat{i}-\hat{j})\times m[\omega\hat{i}\times L(\hat{i}-\hat{j})]$$
$$= L(-\hat{i}+\hat{j})\times m\omega L\hat{k} + L(\hat{i}-\hat{j})\times -m\omega L\hat{k}$$
$$= m\omega L^2(\hat{j}+\hat{i}) + m\omega L^2(\hat{j}+\hat{i})$$

$$\therefore \boxed{\underline{H}_{/o} = 2m\omega L^2(\hat{i}+\hat{j})}$$

$\dot{\underline{H}}_{/o} = \sum_{i=1}^{2} \underline{r}_{i/o} \times m_i \underline{a}_i$ where $\underline{a}_i = \underline{\omega}\times(\underline{\omega}\times\underline{r}_{i/o}) = \underline{\omega}\times\underline{v}_i$

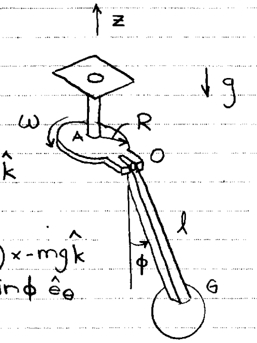
$$= \underline{r}_{1/o}\times m(\underline{\omega}\times\underline{v}_1) + \underline{r}_{2/o}\times m(\underline{\omega}\times\underline{v}_2)$$
$$= L(-\hat{i}+\hat{j})\times m[\omega\hat{i}\times m\omega L\hat{k}]$$
$$+ L(\hat{i}-\hat{j})\times m[\omega\hat{i}\times -m\omega L\hat{k}]$$
$$= L(-\hat{i}+\hat{j})\times -m\omega^2 L\hat{j} + L(\hat{i}-\hat{j})\times m\omega^2 L\hat{j}$$
$$= m\omega^2 L^2\hat{k} + m\omega^2 L^2\hat{k}$$

$$\therefore \boxed{\dot{\underline{H}}_{/o} = 2m\omega^2 L^2\hat{k}}$$

8.26

A turntable rotates @ constant rate, ω, about the z-axis. At the edge swings a pendulum which makes a constant angle ϕ w/ the z-axis. The pendulum consists of a pt. mass connected to a massless rod.

a) Find an equation for ϕ.



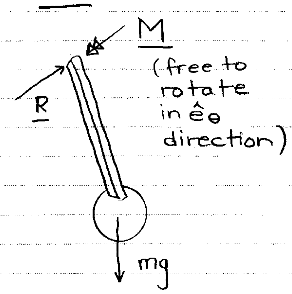
$AMB_{/o}$: $\Sigma\underline{M}_{/o} = \dot{\underline{H}}_{/o}$

$$\Sigma\underline{M}_{/o} = \underline{M} + \underline{r}_{G/o}\times -mg\hat{k}$$
$$= M_r\hat{e}_R + M_z\hat{k}$$
$$+ \ell(\sin\phi\hat{e}_R - \cos\phi\hat{k})\times -mg\hat{k}$$
$$= M_r\hat{e}_R + M_z\hat{k} + mg\ell\sin\phi\,\hat{e}_\theta$$

$\dot{\underline{H}}_{/o} = \underline{r}_{G/o}\times m\underline{a}$

where $\underline{a}_G = \underline{\omega}\times(\underline{\omega}\times\underline{r}_{G/A})$

FBD



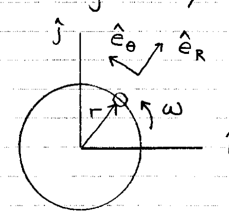
$$\Rightarrow \underline{a}_G = \omega\hat{k}\times[\omega\hat{k}\times(r\hat{e}_R - \ell\cos\phi\hat{k})]$$
$$= \omega\hat{k}\times\omega r\hat{e}_\theta$$
$$= -\omega^2 r\hat{e}_R$$

$$\Rightarrow \dot{\underline{H}}_{/o} = \ell(\sin\phi\hat{e}_R - \cos\phi\hat{k})\times -m\omega^2 r\hat{e}_R$$
$$= m\omega^2\ell r\cos\phi\,\hat{e}_\theta$$

$AMB_{/o}\cdot\hat{e}_\theta$

$\Rightarrow$ $\cancel{m}g\cancel{\ell}\sin\phi = \cancel{m}\omega^2\cancel{\ell}r\cos\phi$

Top view of pt. mass trajectory



$$g\sin\phi = \omega^2 r\cos\phi$$
$$= \omega^2(R+\ell\sin\phi)\cos\phi$$

$$\therefore \boxed{\tan\phi = \frac{\omega^2}{g}(R+\ell\sin\phi)}$$

where $r = R+\ell\sin\phi$

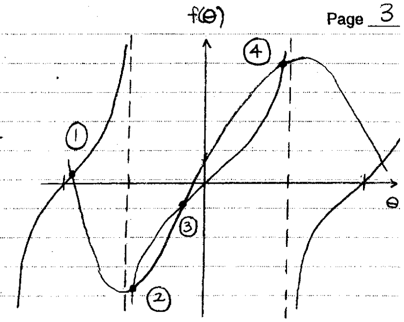
b) Find all possible angles for steady circular motion.

get $\tan\phi = \frac{4}{5}(1+5\sin\phi)$ $\left\{\begin{array}{l}\omega = 2\ \text{rad/s},\ R = 2\text{m} \\ g = 10\ \text{m/s}^2,\ \ell = 10\text{m}\end{array}\right.$

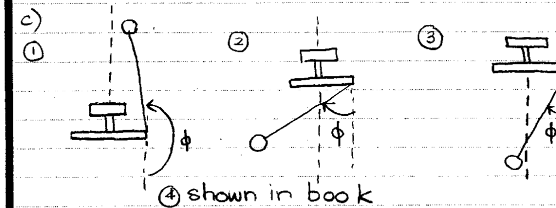
(continued)

8.26 contd.

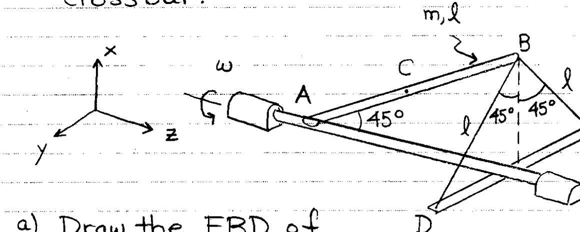
Plots of $\tan\phi$ and $\frac{4}{5}(1+5\sin\phi)$ show 4 intersections $\Rightarrow$ 4 possible angles for steady motion.



c)

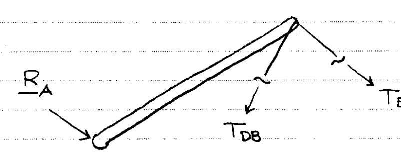


8.44 A rod is attached to a shaft which spins @ constant rate ω about the z-axis. There is a ball and socket joint @ A and B is held by 2 strings connected to a crossbar.



a) Draw the FBD of the rod.

FBD



b) Find T_{DB}.

i) LMB: $\Sigma\underline{F} = m\underline{a}_c$

where $\underline{a}_c = \underline{\omega}\times(\underline{\omega}\times\underline{r}_{c/A})$
$$= \omega\hat{k}\times[\omega\hat{k}\times\tfrac{\ell}{2\sqrt{2}}(\hat{i}+\hat{k})]$$
$$= \omega\hat{k}\times\tfrac{\omega\ell}{2\sqrt{2}}\hat{j}$$

(continued)

$\Rightarrow \underline{a}_c = -\frac{\omega^2 l}{2\sqrt{2}}\,\hat{i}$

$\Sigma \underline{F} = \underline{R}_A + T_{DB}\,\hat{\lambda}_{DB} + T_{EB}\,\hat{\lambda}_{EB}$

where $\hat{\lambda}_{DB} = \frac{\underline{r}_{D/B}}{|\underline{r}_{D/B}|} = \frac{l\hat{j}-l\hat{i}}{l\sqrt{2}} = \frac{1}{\sqrt{2}}(\hat{j}-\hat{i})$

$\hat{\lambda}_{EB} = \frac{\underline{r}_{E/B}}{|\underline{r}_{E/B}|} = \frac{-l\hat{j}-l\hat{i}}{l\sqrt{2}} = \frac{1}{\sqrt{2}}(-\hat{j}-\hat{i})$

- Using LMB, we get 3 equations for 5 unknowns ($\underline{R}_A$, T_{DB}, T_{EB}). So, take AMB about, say, pt. A :

$\underline{\text{AMB}/_A}$: $\Sigma \underline{M}_{/A} = \dot{\underline{H}}_{/A}$

$\Sigma \underline{M}_{/A} = \underline{r}_{B/A} \times T_{DB}\,\hat{\lambda}_{DB} + \underline{r}_{B/A} \times T_{EB}\,\hat{\lambda}_{EB}$

$= \frac{l}{\sqrt{2}}(\hat{k}+\hat{i}) \times \frac{1}{\sqrt{2}}T_{DB}(\hat{j}-\hat{i}) + \frac{l}{\sqrt{2}}(\hat{k}+\hat{i}) \times \frac{1}{\sqrt{2}}T_{EB}(-\hat{j}-\hat{i})$

$= \frac{1}{2}T_{DB}\,l\,(-\hat{i}-\hat{j}+\hat{k}) + \frac{1}{2}T_{EB}\,l\,(\hat{i}-\hat{j}-\hat{k})$

$\dot{\underline{H}}_{/A} = \int \underline{r}_{/A} \times \underline{a}\;dm$

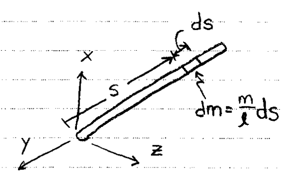
$= \int \left[\frac{1}{\sqrt{2}}s(\hat{k}+\hat{i}) \times \underbrace{-\frac{1}{\sqrt{2}}s\omega^2\,\hat{i}}_{\text{similar to } a_c \text{ above w/ } \frac{l}{2}\mapsto s}\right]\frac{m}{l}\,ds$

$= \left(\int_0^l -\frac{1}{2}\omega^2\frac{m}{l}s^2\,ds\right)\hat{j}$

$= -\frac{m\omega^2}{l}\int_0^l s^2\,ds\;\hat{j}$

$= -\frac{m\omega^2}{l}\left(\frac{1}{3}l^3\right)\hat{j}$

$= -\frac{1}{6}m\omega^2 l^2\,\hat{j}$



$\text{AMB}/_A \cdot \hat{i} \Rightarrow -\frac{1}{2}T_{DB}\,l + \frac{1}{2}T_{EB}\,l = 0 \quad (1)$
($\&$ $\text{AMB}/_A \cdot \hat{k}$)

$\text{AMB}/_A \cdot \hat{j} \Rightarrow -\frac{1}{2}T_{DB}\,l - \frac{1}{2}T_{DB}\,l = -\frac{1}{6}m\omega^2 l^2$

(1) $\Rightarrow T_{DB} = T_{EB}$
(2) $\Rightarrow -T_{DB}\,l = -\frac{1}{6}m\omega^2 l$

$\therefore \boxed{T_{DB} = \frac{1}{6}m\omega^2 l}$

(continued)

ii) $\underline{\text{AMB}}/_{\text{axis AE}}$: $\{\Sigma \underline{M}_{/A} = \dot{\underline{H}}_{/A}\} \cdot \hat{\lambda}_{EA}$

where $\hat{\lambda}_{EA} = \frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$

$\Sigma \underline{M}_{/A} \cdot \hat{\lambda}_{EA} = \frac{1}{2}T_{DB}\frac{l}{\sqrt{2}}(1+1) + \frac{1}{2}T_{EB}\frac{l}{\sqrt{2}}(1-1)$

$= T_{DB}\frac{l}{\sqrt{2}}$

$\dot{\underline{H}}_{/A} \cdot \hat{\lambda}_{EA} = \frac{1}{\sqrt{2}}\frac{1}{6}m\omega^2 l$

$\Rightarrow \frac{l}{\sqrt{2}}T_{DB} = \frac{1}{\sqrt{2}}\frac{1}{6}m\omega^2 l$

$\therefore \boxed{T_{DB} = \frac{1}{6}m\omega^2 l}$ ← • same as above using just one equation

c) Find $\underline{R}_A$ by using LMB in b) i) :

$\underline{R}_A = -T_{DB}\,\hat{\lambda}_{DB} - T_{EB}\,\hat{\lambda}_{EB} + m\underline{a}_c$

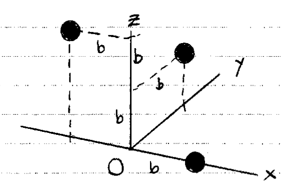
$= -\frac{1}{6}m\omega^2 l\,\frac{1}{\sqrt{2}}(\hat{j}-\hat{i}) - \underbrace{\frac{1}{6}m\omega^2 l}_{\text{from (1)}}\frac{1}{\sqrt{2}}(-\hat{j}-\hat{i})$

$-\frac{m}{2\sqrt{2}}\omega^2 l\,\hat{i}$

$\Rightarrow \boxed{\underline{R}_A = -\frac{\sqrt{2}}{12}m\omega^2 l\,\hat{i}}$

$\underline{8.49}$ Three identical solid spheres are located as shown. Find $[I^O]$.

Strategy :
Find $[I^O]$ for each sphere using $/\!/$-axis thm. and add them up.



$$[I^O] = [I^{CM}] + m\begin{bmatrix} y_{CM/O}^2 + z_{CM/O}^2 & -x_{CM/O}y_{CM/O} & -x_{CM/O}z_{CM/O} \\ -x_{CM/O}y_{CM/O} & x_{CM/O}^2 + z_{CM/O}^2 & -y_{CM/O}z_{CM/O} \\ -x_{CM/O}z_{CM/O} & -y_{CM/O}z_{CM/O} & x_{CM/O}^2 + y_{CM/O}^2 \end{bmatrix}$$

① $$[I^O]_1 = \begin{bmatrix} \frac{2}{5}mR^2 & 0 & 0 \\ 0 & \frac{2}{5}mR^2 & 0 \\ 0 & 0 & \frac{2}{5}mR^2 \end{bmatrix}$$

$$+m\begin{bmatrix} 0^2+(2b)^2 & -(-b)(0) & -(-b)(2b) \\ -(-b)(0) & b^2+(2b)^2 & -(0)(2b) \\ -(-b)(2b) & -(0)(2b) & b^2+0^2 \end{bmatrix}$$

(continued)

$$\Rightarrow [I^O]_1 = m\begin{bmatrix} \frac{2}{5}R^2+4b^2 & 0 & 2b^2 \\ 0 & \frac{2}{5}R^2+5b^2 & 0 \\ 2b^2 & 0 & \frac{2}{5}R^2+b^2 \end{bmatrix}$$

② $$[I^O]_2 = \begin{bmatrix} \frac{2}{5}mR^2 & 0 & 0 \\ 0 & \frac{2}{5}mR^2 & 0 \\ 0 & 0 & \frac{2}{5}mR^2 \end{bmatrix}$$

$$+m\begin{bmatrix} 0^2+0^2 & -(b)(0) & -(b)(0) \\ -(b)(0) & b^2+0^2 & -(0)(0) \\ -(b)(0) & -(0)(0) & b^2+0^2 \end{bmatrix}$$

$$\Rightarrow [I^O]_2 = m\begin{bmatrix} \frac{2}{5}R^2 & 0 & 0 \\ 0 & \frac{2}{5}R^2+b^2 & 0 \\ 0 & 0 & \frac{2}{5}R^2+b^2 \end{bmatrix}$$

③ $$[I^O]_3 = \begin{bmatrix} \frac{2}{5}mR^2 & 0 & 0 \\ 0 & \frac{2}{5}mR^2 & 0 \\ 0 & 0 & \frac{2}{5}mR^2 \end{bmatrix}$$

$$+m\begin{bmatrix} b^2+b^2 & -(0)(b) & -(0)(b) \\ -(0)(b) & 0^2+b^2 & -(b)(b) \\ -(0)(b) & -(b)(b) & 0^2+b^2 \end{bmatrix}$$

$$\Rightarrow [I^O]_3 = m\begin{bmatrix} \frac{2}{5}R^2+2b^2 & 0 & 0 \\ 0 & \frac{2}{5}R^2+b^2 & -b^2 \\ 0 & -b^2 & \frac{2}{5}R^2+b^2 \end{bmatrix}$$

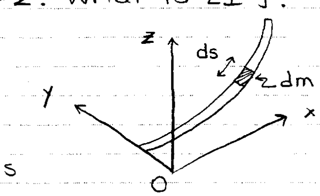
$\therefore [I^O] = [I^O]_1 + [I^O]_2 + [I^O]_3$

$$\boxed{[I^O] = m\begin{bmatrix} \frac{6}{5}R^2+6b^2 & 0 & 2b^2 \\ 0 & \frac{6}{5}R^2+7b^2 & -b^2 \\ 2b^2 & -b^2 & \frac{6}{5}R^2+3b^2 \end{bmatrix}}$$

Problem 5

A curved rod of mass $m = 3\,\text{kg}$ is along the curve $x = \sin t$, $y = e^t$, $z = t^2$ for $0 \le t \le 2$. What is $[I^O]$?

Look at a small segment of the rod :



$dm = \rho\,ds = \frac{m}{l}ds$

where $l = \int_{t=0}^{t=2} ds$ and $ds^2 = dx^2 + dy^2 + dz^2$

(continued)

Note that ds can also be written as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \sqrt{\cos^2 t + e^{2t} + 4t^2}\, dt$$

$$\therefore\ \ell = \int_0^2 \sqrt{\cos^2 t + e^{2t} + 4t^2}\, dt$$

Now get elements of $[I^o]$:

$$I^o_{xx} = \int (y_{/o}^2 + z_{/o}^2)\, dm$$
$$= \rho \int_0^2 (e^{2t}+t^4)\sqrt{\cos^2 t + e^{2t}+4t^2}\, dt$$

$$I^o_{yy} = \int (x_{/o}^2 + z_{/o}^2)\, dm$$
$$= \rho \int_0^2 (\sin^2 t + t^4)\sqrt{}\, dt$$

$$I^o_{zz} = \int (x_{/o}^2 + y_{/o}^2)\, dm$$
$$= \rho \int_0^2 (\sin^2 t + e^{2t})\sqrt{}\, dt$$

$$I^o_{xy} = -\int x_{/o}\, y_{/o}\, dm$$
$$= -\rho \int_0^2 (\sin t) e^t \sqrt{}\, dt$$

$$I^o_{xz} = -\int x_{/o}\, z_{/o}\, dm$$
$$= -\rho \int_0^2 (\sin t) t^2 \sqrt{}\, dt$$

$$I^o_{yz} = -\int y_{/o}\, z_{/o}\, dm$$
$$= -\rho \int_0^2 t^2 e^t \sqrt{}\, dt$$

To calculate these integrals, you can write your own integration routine (e.g. trapezoidal rule, Simpson's rule, etc.) or use Matlab's QUAD or QUAD8 functions.


```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This script file determines the moment of inertia
% matrix for the final problem in HW#9.  Calls are
% made to Matlab's quad8 function which does
% numerical integration of the given function 'func
% from x1 to x2.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

m=3;   % mass of rod in kg

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% To use 'quad8' you must supply a function, i.e.
% the integrand, which must return a vector.  You
% must also give the limits of integration (here we
% integrate from t=0 to t=2).
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

L=quad8('arclength',0,2);  % length of the rod (m)
rho=m/L;                   % rod density (kg/m)

Ixx=rho*quad8('Ixx_func',0,2);
Ixy=rho*quad8('Ixy_func',0,2);
Ixz=rho*quad8('Ixz_func',0,2);
Iyy=rho*quad8('Iyy_func',0,2);
Iyz=rho*quad8('Iyz_func',0,2);
Izz=rho*quad8('Izz_func',0,2);

I=[Ixx Ixy Ixz;
   Ixy Iyy Iyz;
   Ixz Iyz Izz]

% Answer:
% [I]=[ 76.8631  -11.2810   -5.5291
%       -11.2810  18.1371  -30.8026
%        -5.5291 -30.8026   63.2581 ]  kg m^2

% The following are separate function files:

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=arclength(t)
f=sqrt(cos(t).^2+exp(2*t)+4*t.^2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Ixx_func(t)
f=(exp(2*t)+t.^4).*sqrt(cos(t).^2+exp(2*t)+4*t.^2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Ixy_func(t)
f=-sin(t).*exp(t).*sqrt(cos(t).^2+exp(2*t)+4*t.^2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Ixz_func(t)
f=-sin(t).*t.^2.*sqrt(cos(t).^2+exp(2*t)+4*t.^2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Iyy_func(t)
f=(sin(t).^2+t.^4).*sqrt(cos(t).^2+exp(2*t)+4*t.^2)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Iyz_func(t)
f=-exp(t).*t.^2.*sqrt(cos(t).^2+exp(2*t)+4*t.^2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=Izz_func(t)
f=(exp(2*t)+sin(t).^2).*sqrt(cos(t).^2+exp(2*t)+
   4*t.^2);
```