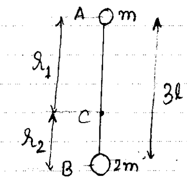
HW #8 T&AM 203

DUE: MARCH 28^{th} '00 – PRITAM GANGULY

Notation "MI" stands for moment of Inertia

#(7.106)



Think first & then calculate !!

a) Which point is I_{zz} minimum (A, B, C) ??

For a planar rigid body I_{zz}^{CM} should be the least as I_{zz} about any other point can be written as $I_{zz}^{O} = I_{zz}^{CM} + md^2$

$\therefore I_{zz}^{O} > I_{zz}^{CM}$

Calculation: $\boxed{I_{zz}^{A} = (2m)\cdot(3l)^2 = 18ml^2}$

$\boxed{I_{zz}^{B} = (m)\cdot(3l)^2 = 9ml^2}$

Since, C is the CM, $mr_1 = 2mr_2$

$\Rightarrow \boxed{r_1 = 2r_2}$ & $r_1 + r_2 = 3l$

$\Rightarrow r_1 = 2l \quad r_2 = l$

$\therefore \boxed{I_{zz}^{C} = m(2l)^2 + 2m(l)^2 = 6ml^2}$

$\therefore I_{zz}^{C}$ is minimum

b) Which is maximum $I_{zz}^{A}, I_{zz}^{B}, I_{zz}^{C}$?

I_{zz}^{C} is already the minimum

Between I_{zz}^{A} and I_{zz}^{B}, the arm length is same for both ($3l$) but the mass that rotates about axis through A is larger ($=2m$)

So I_{zz}^{A} should be largest

Calculation: See from (a) that I_{zz}^{A} is the largest among $I_{zz}^{A}, I_{zz}^{B}, I_{zz}^{C}$

(c) Ratio of I_{zz}^{A} to I_{zz}^{B} ??

Since the arm length of the masses from their respective axes are same in both cases, the ratio between the MI's is nothing but ratio of the masses rotating about the corresponding axes.

$$\boxed{\frac{I_{zz}^{A}}{I_{zz}^{B}} = 2}$$

contd →

(d) Radius of Gyration (r_G)? & is it less than $3l$??

Radius of gyration is the radius of a hoop of equivalent mass centered at the Center of Mass of the original system & having same MI too !!

So since both masses are @ a distance $< 3l$ from CM, a hoop formed which has equal mass distribution around it has radius $< 3l$

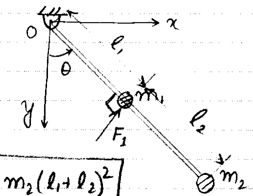
Calculation –

$(m+2m)r_G^2 = I_{zz}^{C} = 6ml^2$

$\Rightarrow \boxed{r_G = \sqrt{2}\,l}$

it can be thus seen that $\boxed{r_G < 3l}$

#(7.121)



a) $I_{zz}^{O} = ?$

$\boxed{I_{zz}^{O} = m_1 l_1^2 + m_2(l_1+l_2)^2}$

b) $\vec{H}_O = ?$

Since point O is fixed

$\vec{H}_O = I_{zz}^{O}\cdot\vec{\omega} \qquad \vec{\omega} = \dot{\theta}(-\hat{k})$

$\Rightarrow \boxed{\vec{H}_O = -I_{zz}^{O}\dot{\theta}\hat{k} = -\{m_1l_1^2 + m_2(l_1+l_2)^2\}\dot{\theta}\hat{k}}$

c) $\dot{\vec{H}}_O = ?$

again since point O is fixed & it is fixed axis rotation

$\dot{\vec{H}}_O = I_{zz}^{O}\ddot{\theta}(-\hat{k})$

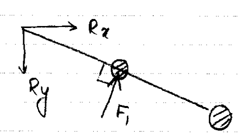
$\Rightarrow \boxed{\dot{\vec{H}}_O = -\{m_1l_1^2 + m_2(l_1+l_2)^2\}\ddot{\theta}\hat{k}}$

(d) Kinetic Energy $E_K = ?$

$\boxed{E_K = \frac{1}{2}I_{zz}^{O}\omega^2 = \frac{1}{2}\{m_1l_1^2 + m_2(l_1+l_2)^2\}\dot{\theta}^2}$

(e) Don't know $\theta, \dot{\theta}, \ddot{\theta}$, know F_1 ($\perp$ to rod) $\ddot{\theta} = ?$ (neglect gravity)

FBD –



contd →

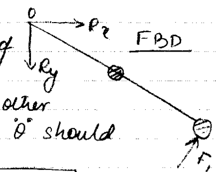
(e) contd

AMB about O gives,

$\vec{M}_O = -F_1 l_1\hat{k} = \dot{\vec{H}}_O = -I_{zz}^{O}\ddot{\theta}\hat{k}$

$$\Rightarrow \boxed{\ddot{\theta} = \frac{F_1 l_1}{I_{zz}^{O}} = \frac{F_1 l_1}{\{m_1l_1^2 + m_2(l_1+l_2)^2\}}}$$

f) $\ddot{\theta}$ bigger or smaller if F_1 applied to m_2?

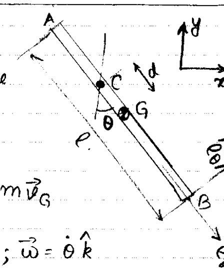


If F_1 is applied to m_2 instead of m_1, then the moment arm becomes (l_1+l_2) about O. The other quantities remain same so $\ddot{\theta}$ should increase

$$\boxed{\ddot{\theta} = \frac{F_1(l_1+l_2)}{\{m_1l_1^2 + m_2(l_1+l_2)^2\}}}$$

#(7.128) Pegged pendulum

pegged @ C, G is CM of the whole rod



a) $\vec{H}_C = ?$

$\vec{H}_C = I_{zz}^{G}\vec{\omega} + \vec{r}_{G/C}\times m\vec{v}_G$

$\vec{v}_G = d\dot{\theta}\hat{e}_\theta \;;\; \vec{r}_{G/C} = d\hat{e}_r \;;\; \vec{\omega} = \dot{\theta}\hat{k}$

$\Rightarrow \vec{H}_C = I_{zz}^{G}\dot{\theta}\hat{k} + m(d\hat{e}_r \times d\dot{\theta}\hat{e}_\theta) = \underbrace{(I_{zz}^{G} + md^2)}_{I_{zz}^{C}}\dot{\theta}\hat{k}$

$I_{zz}^{G} = \frac{ml^2}{12}$

$\therefore \boxed{\vec{H}_C = m(l^2/12 + d^2)\dot{\theta}\hat{k}}$

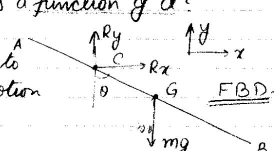
b) $\dot{\vec{H}}_C = ?$

$\dot{\vec{H}}_C = \frac{d}{dt}\left[\left(\frac{ml^2}{12}+md^2\right)\dot{\theta}\hat{k}\right]$

$\boxed{\Rightarrow \dot{\vec{H}}_C = \left(\frac{ml^2}{12}+md^2\right)\ddot{\theta}\hat{k}}$ ($\because \hat{k}$ is invariant with time in this case)

c) To find the period as a function of d?

To find the period, its best to start with equation of motion



AMB about C

$\vec{M}_C = -mg\sin\theta\, d\,\hat{k} = \dot{\vec{H}}_C = \left(\frac{ml^2}{12}+md^2\right)\ddot{\theta}\hat{k}$

$\Rightarrow \boxed{\ddot{\theta} + \frac{gd}{(l^2/12 + d^2)}\sin\theta = 0}$ (1)

contd →

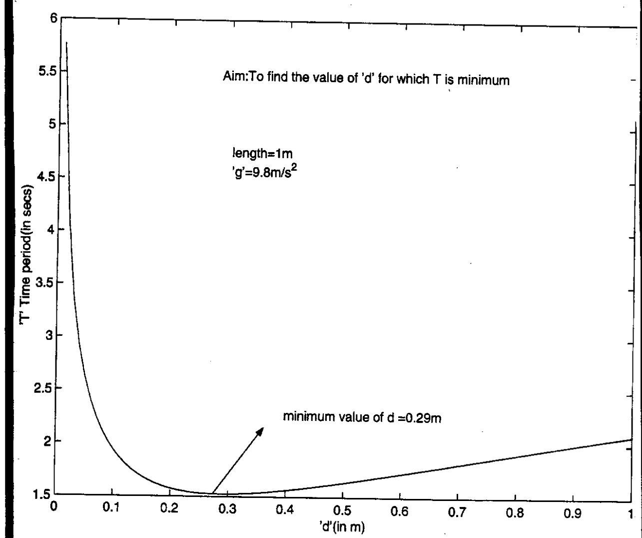
(c) contd Making small angle approximations
i.e. $\sin\theta \approx \theta$

Eqn ① becomes $\boxed{\ddot{\theta} + \dfrac{gd}{(\ell^2/12 + d^2)}\,\theta = 0}$ (This represents the SHM equation)

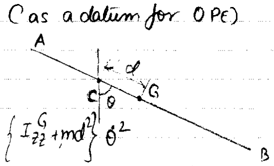
$\therefore\ T = 2\pi\sqrt{\dfrac{\ell^2/12 + d^2}{gd}}$ Dividing numerator & denominator by ℓ^2 & rearranging

$$\boxed{T = 2\pi\sqrt{\frac{\ell}{g}}\sqrt{\frac{1}{12(d/\ell)} + (d/\ell)}}$$

To make a plot choose $\ell = 1m;\ g = 9.8\ m/s^2$



(d) Total energy ?? (choose C as a datum for O PE)



$E_P = -mgd\cos\theta$

$E_K = \frac{1}{2}I_{zz}^G\dot{\theta}^2 + \frac{1}{2}mv_G^2 = \frac{1}{2}\{I_{zz}^G + md^2\}\dot{\theta}^2$ (since $v_G = d\dot{\theta}$).

$\therefore\ E_K = \frac{1}{2}I_{zz}^C\dot{\theta}^2$

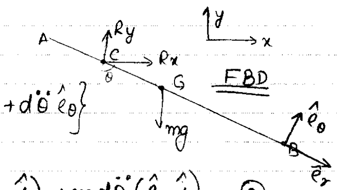
$$\boxed{\text{Total Energy} = E_P + E_K = -mgd\cos\theta + \tfrac{1}{2}m\left(\tfrac{\ell^2}{12} + d^2\right)\dot{\theta}^2}$$

contd →

(e) @ $\theta = \pi/6$ $\ddot{\theta} = ?$
from equation ①. $\ddot{\theta} = \dfrac{-gd\sin\theta}{\frac{\ell^2}{12} + d^2}$

@ $\theta = \pi/6$ $\boxed{\ddot{\theta} = \dfrac{-6gd}{\ell^2 + 12d^2}}$

(f) Reactions @ C ??



LMB ÷

$-mg\hat{j} + R_y\hat{j} + R_x\hat{i} = m\{-d\dot{\theta}^2\hat{e}_r + d\ddot{\theta}\hat{e}_\theta\}$

Dotting with $\hat{i}$
$R_x = -md\dot{\theta}^2(\hat{e}_r\cdot\hat{i}) + md\ddot{\theta}(\hat{e}_\theta\cdot\hat{i})$ — ②

Dotting with $\hat{j}$
$R_y = mg - md\dot{\theta}^2(\hat{e}_r\cdot\hat{j}) + md\ddot{\theta}(\hat{e}_\theta\cdot\hat{j})$ — ③

Now,
$\hat{e}_r\cdot\hat{i} = \sin\theta$ $\hat{e}_r\cdot\hat{j} = -\cos\theta$ $\hat{e}_\theta\cdot\hat{i} = \cos\theta$ $\hat{e}_\theta\cdot\hat{j} = \sin\theta$
Using these and equation ① in eqns ② & ③

we get $R_x = -md\dot{\theta}^2\sin\theta + md\left(\dfrac{-g\sin\theta\, d}{\ell^2/12 + d^2}\right)\cos\theta$

$R_y = mg + md\dot{\theta}^2\cos\theta + md\left(\dfrac{-g\sin\theta d}{\frac{\ell^2}{12} + d^2}\right)\sin\theta$

i.e. $\boxed{R_x = -m\left(d\dot{\theta}^2\sin\theta + \dfrac{gd^2\sin\theta\cos\theta}{(\ell^2/12 + d^2)}\right)}$

$\boxed{R_y = m\left(g + d\dot{\theta}^2\cos\theta - \dfrac{gd^2\sin^2\theta}{(\ell^2/12 + d^2)}\right)}$

(g) for a given ℓ, for what value of d is T smallest

We found $T = \underbrace{2\pi\sqrt{\ell/g}}_{\text{constant}}\sqrt{\dfrac{1}{12(d/\ell)} + (d/\ell)}$

To minimise $\dfrac{1}{12(d/\ell)} + (d/\ell) = K$ (from high school we know $\frac{a+b}{2} \geq \sqrt{ab}$ & the equality occurs $a = b$)

Method 1 ÷
So minimum value of K is when $\dfrac{1}{12(d/\ell)} = d/\ell \Rightarrow (d/\ell) = \sqrt{\dfrac{1}{12}}$

$\Rightarrow \boxed{d = 0.29\ell}$

Method 2 ÷
let $d/\ell = x$

$K = \dfrac{1}{12x} + x$ $\dfrac{dK}{dx} = -\dfrac{1}{12x^2} + 1 = 0 \Rightarrow x = \sqrt{\dfrac{1}{12}}$

$\Rightarrow \boxed{d = 0.29\ell}$

→ contd.

(h) To find error in measurement of T when d increases due to wear and tear
$\ell = 1m$

(i) d varies $0.15m \to 0.16m$
$T_{initial} = 2\pi\sqrt{\dfrac{1}{g}}\sqrt{\dfrac{1}{(0.15)\times 12} + (0.15)} \approx 1.686$ sec

$T_{final} = 2\pi\sqrt{\dfrac{1}{g}}\sqrt{\dfrac{1}{(0.16)\times 12} + 0.16} = 1.656$ sec

$$\boxed{\text{error} = \frac{(T_{final} - T_{initial})}{T_{initial}}\times 100 = 1.78\%}$$

(ii) $d = 0.29m \to 0.30m$ $T_{initial} = 1.52506$ sec
$T_{final} = 1.52562$ sec

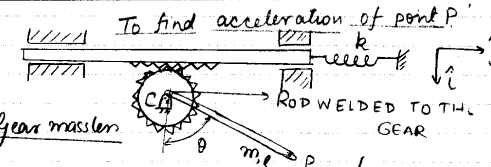
$\boxed{\text{error} = 0.037\%}$

(iii) $d = 0.45 \to 0.46m$ $T_{initial} = 1.5996$ sec
$T_{final} = 1.6071$ sec

$\boxed{\text{error} = 0.47\%}$

We see that minimum error occurs when $d = 0.29\ell$, this was the minimum of T we found before. We know that for a smooth function, the variation around the maximum or minimum is the least & in this case it is clearly evident from the plot that for the same change in d the change in T is least around $d = 0.29\ell$.

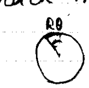
#7.148 To find acceleration of point P



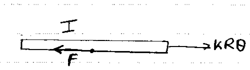
Rack & Gear massless

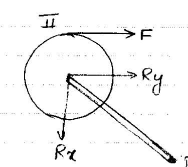
Point C is fixed in space. (NEGLECT GRAVITY)

Kinematic Constraint ÷ If the gear rotates by θ (& so does the rod) counterclockwise then the rack moves to the left by Rθ



FBDs





Since the rack is massless
LMB ÷ $(-F + KR\theta)\hat{j} = 0$
$\Rightarrow F = KR\theta$ — ①

contd →

7.148 contd

for the FBD II, writing AMB about C

$\vec{M}_C = -FR\hat{k} = \dot{\vec{H}}_C = I^C_{zz}\ddot{\theta}\hat{k}$ (Gear is massless)

$I^C_{zz} = \frac{ml^2}{3}$

$\therefore \left\{-FR\hat{k} = \frac{ml^2}{3}\ddot{\theta}\hat{k}\right\}\cdot\hat{k}$

$\Rightarrow \ddot{\theta} + \frac{3FR}{ml^2} = 0$ — ②

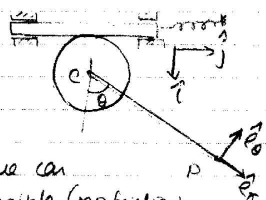
Substituting for F from eqn ①

$\boxed{\ddot{\theta} + \left(\frac{3kR^2}{ml^2}\right)\theta = 0}$ — ③

Now acceleration of point P is given by

$\boxed{\vec{a}_P = -l\dot{\theta}^2\hat{e}_r + l\ddot{\theta}\hat{e}_\theta}$ — ④

(∵ P rotates in circles about the fixed point C)



To find $\dot{\theta}$ as a function of θ, we can use conservation of energy principle (no friction)

assuming C as a datum for 0 PE

@ $t=0$, $\theta=\theta_0$, $\dot{\theta}=0$

So $\underbrace{\frac{1}{2}k(R\theta_0)^2}_{\text{Initial Energy}} = \underbrace{\frac{1}{2}k(R\theta)^2 + \frac{1}{2}I^C_{zz}(\dot{\theta})^2}_{\text{Energy @ any }\theta}$

$\therefore \boxed{(\dot{\theta})^2 = \frac{kR^2(\theta_0^2-\theta^2)}{I^C_{zz}} = \frac{3kR^2}{ml^2}(\theta_0^2-\theta^2)}$ — ⑤

Substituting ③ & ⑤ in ④

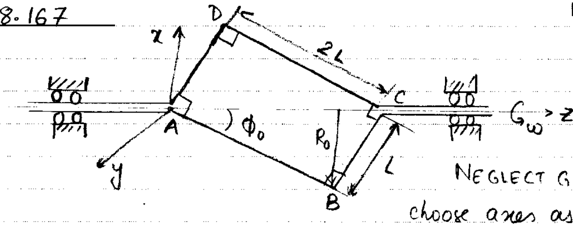
$\vec{a}_P = -\frac{3kR^2}{ml}\left[(\theta_0^2-\theta^2)\hat{e}_r - \theta\hat{e}_\theta\right]$

since we want acceleration of P @ $\theta=0$

so @ $\theta=0$ $\hat{e}_r = \hat{i}$, $\hat{e}_\theta = \hat{j}$.

$\boxed{\vec{a}_P = \frac{-3kR^2\theta_0^2}{ml}\hat{i}}$

8.167



NEGLECT GRAVITY

choose axes as shown origin @ A

a) Velocity of B?

$\vec{v}_A = 0(\hat{i}+\hat{j}+\hat{k})$ (∵ A is on the axis)

$\vec{v}_{B/A} = \vec{\omega}\times\vec{r}_{B/A}$ (∵ A & B are points on same rigid body)

$\vec{r}_{B/A} = (-2L\sin\phi_0\hat{i} + 2L\cos\phi_0\hat{k})$ $\quad \vec{\omega} = \omega\hat{k}$

$\therefore \vec{v}_{B/A} = \omega\hat{k}\times(-2L\sin\phi_0\hat{i} + 2L\cos\phi_0\hat{k})$

$= -2L\omega\sin\phi_0\hat{j}$

$\tan\phi_0 = \frac{1}{2} \Rightarrow \sin\phi_0 = \frac{1}{\sqrt{5}}$

$\therefore \boxed{\vec{v}_{B/A} = -\frac{2L}{\sqrt{5}}\omega\hat{j}}$

(A is on axis)

$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A = \vec{\omega}\times(\vec{\omega}\times\vec{r}_{B/A}) + \underbrace{\cancel{\dot{\vec{\omega}}}\times\vec{r}_{B/A}}_{=0\ (\vec{\omega}=\text{constant})} - \overset{=0}{\cancel{\vec{a}_A}}$

$\therefore \vec{a}_{B/A} = \vec{a}_B = \omega^2\hat{k}\times(-2L\sin\phi_0\hat{j})$

$\Rightarrow \boxed{\vec{a}_B = \frac{2L}{\sqrt{5}}\omega^2\hat{i}}$

Instead of doing all this it is easy to see that B rotates in a circle of radius

$R_0 = \frac{2L}{\sqrt{5}}$ & the above expressions are standard

b) when B is in y-z plane (+ve part of it)

$\vec{r}_{B/A} = (2L\sin\phi_0\hat{j} + 2L\cos\phi_0\hat{k})$

$\boxed{\vec{v}_{B/A} = \vec{v}_B = \omega\hat{k}\times\vec{r}_{B/A} = -\frac{2L\omega}{\sqrt{5}}\hat{i}}$

$\boxed{\vec{a}_{B/A} = \vec{a}_B = \omega^2\hat{k}\times(-2L\sin\phi_0\hat{i}) = -\frac{2L}{\sqrt{5}}\omega^2\hat{j}}$

X —— X.