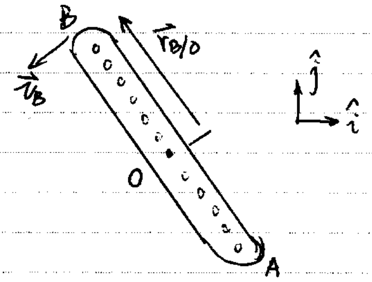
HW #7 — Solutions

7.57



Suppose the rod is pegged at point O, and the angular velocity of the rod is $\omega\hat{k}$.

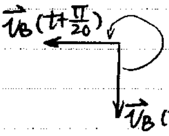
The distance $|OB|$ gives the location of the peg.

From $\vec{v} = \vec{\omega}\times\vec{r}$

thus, $\vec{v}_B = \vec{\omega}\times\vec{r}_{B/O} = \omega\hat{k}\times\vec{r}_{B/O}$

Since, at some instant t, $\vec{v}_B = -3\,\text{m/s}\,\hat{j}$

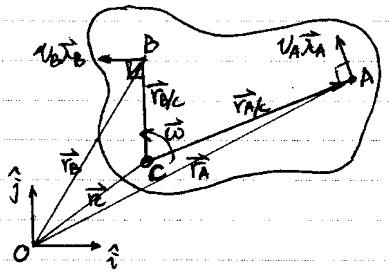
After $\frac{\pi}{20}$ s, $\vec{v}_B = -3\,\text{m/s}\,\hat{i}$



$\therefore \boxed{\omega = \frac{\Delta\theta}{\Delta t} = \frac{\frac{3}{2}\pi}{\frac{\pi}{20}} = 30\ \text{rad/s}}$

$|v_B| = \omega\cdot|OB| \Rightarrow \boxed{|OB| = \frac{|v_B|}{\omega} = \frac{3}{30} = 0.1\,m}$

7.66.



planar motion
$\vec{r}_A, \vec{r}_B, \vec{\lambda}_A, \vec{\lambda}_B$ given

a). As shown in figure, suppose the angular velocity is $\omega\hat{k}$. Since it's a rigid body's circular motion, $\vec{\omega}$ is same for whole body.

$v_A\hat{\lambda}_A = \underline{\omega}\times(\underline{r}_A - \underline{r}_C)$

$v_B\hat{\lambda}_B = \underline{\omega}\times(\underline{r}_B - \underline{r}_C)$

4 scalar eqns but 5 unknowns $v_A, v_B, \underline{r}_C, \omega$

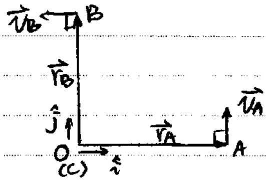
But we're asked only to find $\underline{r}_C$, thus

$$\boxed{\begin{aligned} \frac{v_A}{\omega}\hat{\lambda}_A &= \hat{k}\times(\underline{r}_A - \underline{r}_C) \\ \frac{v_B}{\omega}\hat{\lambda}_B &= \hat{k}\times(\underline{r}_B - \underline{r}_C) \end{aligned}}$$

Now 4 unknowns, so the solution to these will yield $\underline{r}_C$.

Illustration:

1) $\vec{r}_A = 1\hat{i}$, $\vec{r}_B = 1\hat{j}$, $\vec{\lambda}_A = 1\hat{j}$, $\vec{\lambda}_B = -1\hat{i}$, $\vec{r}_C = \vec{0}$

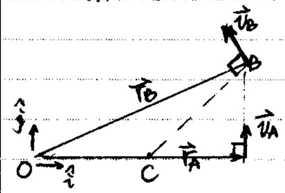


$$\hat{\lambda}_A = \frac{\hat{k}\times(1\hat{i}-\vec{0})}{\|1\hat{i}-\vec{0}\|} = 1\hat{j}$$

$$\vec{\lambda}_B = \frac{\hat{k}\times(1\hat{j}-\vec{0})}{\|1\hat{j}-\vec{0}\|} = -1\hat{i}$$

verified.

2) $\vec{r}_A = 2\hat{i}$, $\vec{r}_B = 2\hat{i}+1\hat{j}$, $\vec{\lambda}_A = 1\hat{j}$, $\hat{\lambda}_B = -\frac{\sqrt{2}}{2}\hat{i}+\frac{\sqrt{2}}{2}\hat{j}$. $\vec{r}_C = 1\hat{i}$



$$\hat{\lambda}_A = \frac{\hat{k}\times(2\hat{i}-1\hat{i})}{\|2\hat{i}-1\hat{i}\|} = \hat{j}$$

$$\hat{\lambda}_B = \frac{\hat{k}\times(2\hat{i}+1\hat{j}-1\hat{i})}{\|2\hat{i}+1\hat{j}-1\hat{i}\|} = -\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}$$

verified.

b) Point C is located where the lines perpendicular to $\hat{\lambda}_A$ & $\hat{\lambda}_B$ cross.

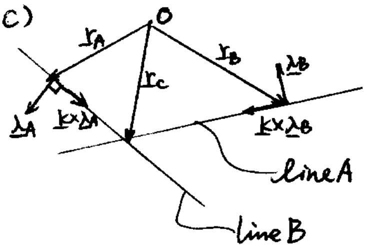
$r_{B/C}\cdot\hat{\lambda}_B = 0 \Rightarrow (r_B - r_C)\cdot\hat{\lambda}_B = 0$

$r_{A/C}\cdot\hat{\lambda}_A = 0 \qquad (r_A - r_C)\cdot\hat{\lambda}_A = 0$

$$\Rightarrow \boxed{\begin{aligned} r_C\cdot\hat{\lambda}_B &= r_B\cdot\hat{\lambda}_B \\ r_C\cdot\hat{\lambda}_A &= r_A\cdot\hat{\lambda}_A \end{aligned}}$$

```
%Problem 7.66 b)

%input four given vectors
%should be in a form [x;y]
r_A=input('enter r_A coordinates:  ');
lambda_A=input('enter lambda_A coordinates:  ');
lambda_A=lambda_A';
r_B=input('enter r_B coordinates:  ');
lambda_B=input('enter lambda_B coordinates:  ');
lambda_B=lambda_B';

%Derived from the governing Eqn
A=[ lambda_A;lambda_B];
b=[lambda_A*r_A ;lambda_B*r_B];
x=A\b;                    % solve linear eqn
disp('r_C is :');
disp(x');
```

c)



parametric eqn. of line A

$$r_1 = r_A + s(k\times\lambda_A)$$

parametric eqn. of line B

$$r_2 = r_B + t(k\times\lambda_B)$$

At intersection at C: $\underline{r}_1 = \underline{r}_2$

$$\{ \underline{r}_A + s(\underline{k}\times \underline{\lambda}_A) = \underline{r}_B + t(\underline{k}\times\underline{\lambda}_B)\}$$

Solve for either s or t

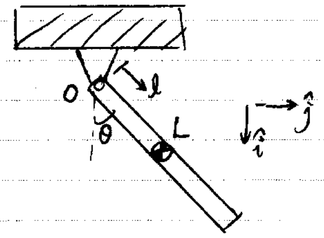
$$\{\ \}\cdot\underline{\lambda}_B \Rightarrow \underline{r}_A\cdot\underline{\lambda}_B + s\,\underline{\lambda}_B\cdot(\underline{k}\times\underline{\lambda}_A) = \underline{\lambda}_B\cdot\underline{r}_B$$

$$\Rightarrow s = \frac{\underline{\lambda}_B\cdot(\underline{r}_B-\underline{r}_A)}{\underline{\lambda}_B\cdot(\underline{k}\times\underline{\lambda}_A)} = \frac{\underline{\lambda}_B\cdot(\underline{r}_B-\underline{r}_A)}{\underline{k}\cdot(\underline{\lambda}_A\times\underline{\lambda}_B)}$$

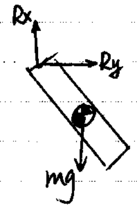
$$\Rightarrow \underline{r}_C = \underline{r}_A + s\,\underline{k}\times\underline{\lambda}_A$$

$$\underline{r}_C = \underline{r}_A + \frac{\overbrace{\underline{\lambda}_A\cdot(\underline{r}_B-\underline{r}_A)}^{\text{proj. of } \underline{r}_{B/A} \text{ in } \underline{\lambda}_A \text{ direction}}}{\underbrace{\underline{k}\cdot(\underline{\lambda}_A\times\underline{\lambda}_B)}_{\sin(\text{angle bet. } \underline{\lambda}_A \,\&\, \underline{\lambda}_B)}}\ \underbrace{\underline{k}\times\underline{\lambda}_A}_{\text{unit vector} \perp \text{ to } \underline{\lambda}_A}$$

$\boxed{7.92}$



a) FBD:



b) AMB: $\boxed{\Sigma\vec{M}_{/O} = \dot{\vec{H}}_{/O}}$

c) left-hand-side:

$$\Sigma\vec{M}_{/O} = \vec{r}_{G/O}\times mg(+\hat{i})$$

$$= \frac{L}{2}(\cos\theta\,\hat{i}+\sin\theta\,\hat{j})\times mg\,\hat{i}$$

$$= \boxed{-\frac{1}{2}mgL\sin\theta\,\hat{k}}$$

d) right-hand-side:

$$\dot{\vec{H}}_{/O} = \int_0^L \vec{r}_{/O}\times\vec{a}\,\frac{m}{L}\,dl$$

$$= \frac{m}{L}\int_0^L l(\cos\theta\,\hat{i}+\sin\theta\,\hat{j})\times[-l\dot{\theta}^2(\cos\theta\,\hat{i}+\sin\theta\,\hat{j}) + l\ddot{\theta}(\cos\theta\,\hat{j}-\sin\theta\,\hat{i})]\,dl$$

$$= \frac{m}{l}\int_0^L [\cancel{-l^2\dot{\theta}^2\cos\theta\sin\theta\,\hat{k}} + l^2\ddot{\theta}\cos^2\theta\,\hat{k} + \cancel{l^2\dot{\theta}^2\sin\theta\cos\theta\,\hat{k}} + l^2\ddot{\theta}\sin^2\theta\,\hat{k}]\,dl$$

$$\vec{H}_{/O} = \frac{m}{L}\int_0^L l^2\ddot{\theta}\, dl\, \hat{k}$$

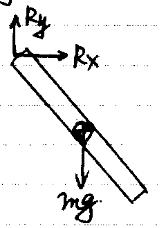
$$= \frac{m}{L}\cdot\frac{L^3}{3}\ddot{\theta}\,\hat{k}$$

$$= \boxed{\frac{m}{3}L^2\ddot{\theta}\,\hat{k}}$$

7.93

a) Swinging rod with mass

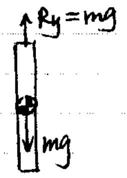
FBD:



It's not a two force member because the body has mass and accelerates.

b) Stationary rod with mass

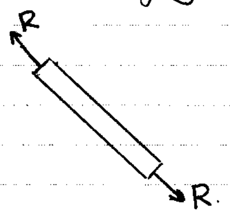
FBD:

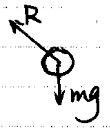


It's a two force member. — the forces are equal and opposite.

c) Massless Swinging rod.

FBD:

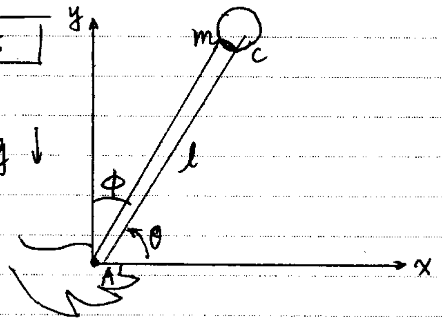




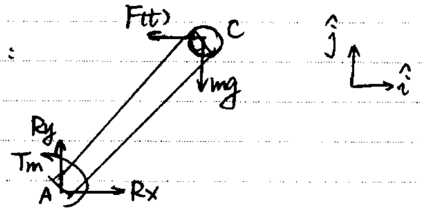
If one includes the ball, the system does not form a two force member (like part a).

If one looks only at the rod (massless), it's a two force member.

$\boxed{7.48}$



a) FBD:



b) AMB $\qquad \Sigma \vec{M}_{/A} = \dot{\vec{H}}_{/A}$

$$\Sigma \vec{M}_{/A} = \vec{r}_{C/A} \times (-mg\hat{j}) + \vec{r}_{C/A} \times F(t)(-\hat{i}) + \vec{T}_m$$

$$= l(\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) \times (-mg\hat{j}) + l(\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) \times F(t)(-\hat{i}) + T_m\hat{k}$$

$$= -mgl\sin\phi\,\hat{k} + F(t)l\cos\phi\,\hat{k} + T_m\hat{k}$$

$$\dot{\vec{H}}_{/A} = \Sigma\, \vec{r}_{/A} \times \vec{a}_c m_c$$

$$= l(\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) \times [l\dot{\phi}^2(-\sin\phi\,\hat{i} - \cos\phi\,\hat{j}) - l\ddot{\phi}(\sin\phi\,\hat{j} - \cos\phi\,\hat{i})]m$$

$$= l^2\dot{\phi}^2(-\sin\phi\cos\phi + \sin\phi\cos\phi)\hat{k}m - ml^2\ddot{\phi}(\sin^2\phi + \cos^2\phi)\hat{k}$$

$$= -ml^2\ddot{\phi}\,\hat{k}$$

$\therefore \{\ \}\cdot\hat{k}:\quad ml^2\ddot{\phi} = mgl\sin\phi - Fl\cos\phi - T_m$

$$\therefore \boxed{\ddot{\phi} - \frac{g}{l}\sin\phi + \frac{F}{ml}\cos\phi + \frac{T_m}{ml^2} = 0} \qquad (*)$$

c) Say, T_m can be any function of ϕ and $\dot{\phi}$

Guess, T_m should be in such a form that makes equation (*) looked like:

$$\ddot{\phi} + c\dot{\phi} + k\phi = 0$$

This is the equation of a damped vibration, and so the pendulum should be driven to $\phi = 0$, stay upright

Alternatively, you can choose ~~the~~ torque T_m to make equation (*) even simpler: $\frac{T}{ml^2} = \frac{g}{l}\sin\phi + c\dot{\phi}$

Then eqn (*) looks like:

$$\ddot{\phi} + c\dot{\phi} = 0 \qquad \text{(Assume } F(t) = 0\text{)}$$

d). Say, $T_m = ml^2(c\dot{\phi} + k\phi)$ $\quad c > 0,\ k > 0$, constant.

$\therefore \ \ddot{\phi} - \frac{g}{l}\sin\phi + c\dot{\phi} + k\phi = 0$

linearize the eqn: when small angle, $\sin\phi \approx \phi$

$\therefore \ \ddot{\phi} + c\dot{\phi} + (k - \frac{g}{l})\phi = 0$

we assume the solution of the form

$\phi = Ae^{\lambda t}$

$\therefore \ \lambda^2 + \lambda c + (k - \frac{g}{l}) = 0$

$\therefore \ \lambda = \frac{-c \pm \sqrt{c^2 - 4(k - \frac{g}{l})}}{2}$

If $c^2 > 4(k - \frac{g}{l})$ $\quad \phi = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$ where $\lambda_1 < 0, \lambda_2 < 0$

If $c^2 = 4(k - \frac{g}{l})$ $\quad \phi = (A_1 + A_2 t) e^{-\frac{c}{2}t}$

If $c^2 < 4(k - \frac{g}{l})$ $\quad \phi = C\sin(\sqrt{4(k - \frac{g}{l}) - c^2}\, t + \psi) e^{-\frac{c}{2}t}$

All of them show that ϕ will decay finally.

e) For the nonlinear equations,

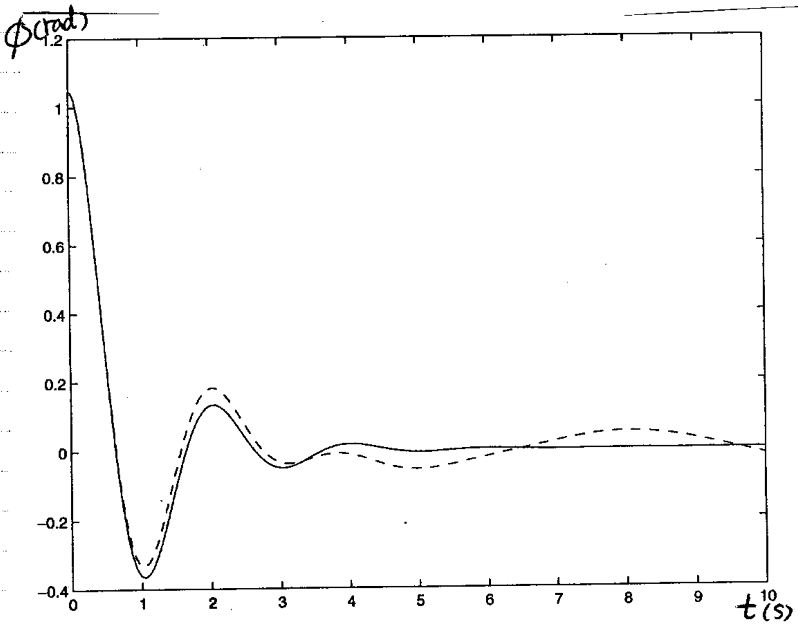
$\ddot{\phi} - \frac{g}{l}\sin\phi + c\dot{\phi} + k\phi = 0$

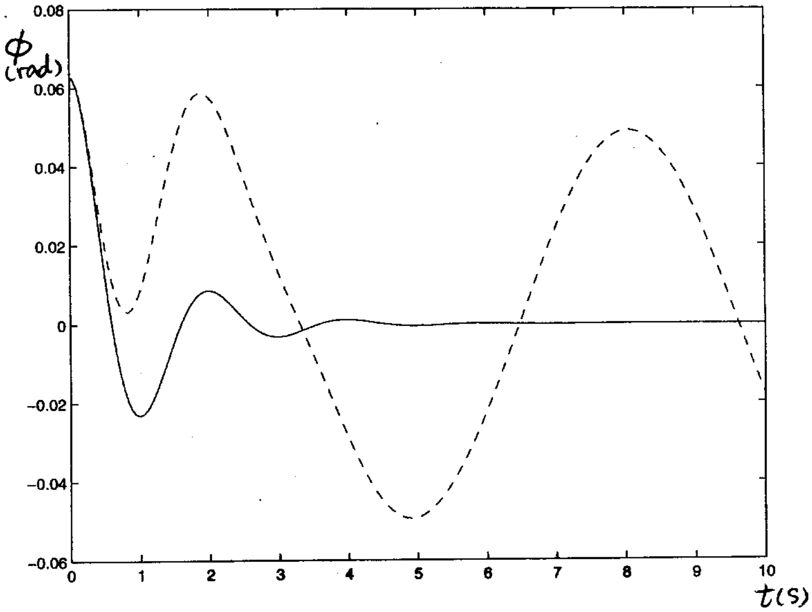
we implement a matlab program to see how the angle ϕ is changed with time.

```
% program for Problem 7.98 e)

global g L m c k ;

g=10; L=1; m=1;c=2;k=20;
tspan=[0 10];
Fi0=pi/3;Fidot0=0;  % initial conditions
z0=[Fi0 Fidot0 ];

[t,z]=ode45('deriv11',tspan,z0);
Fi=z(:,1); Fidot=z(:,2);
plot(t,Fi);
hold on;

[t,z]=ode45('deriv12',tspan,z0);
fi1=z(:,1); fidot=z(:,2);
plot(t,fi1,'--');
```

```
function zdot=deriv11(t,z);

global g L m c k;

Fi=z(1);
alpha=z(2);
alphadot=+g/L*sin(Fi)-c*alpha-k*Fi;

zdot=[alpha alphadot]';
```

```
function zdot=deriv12(t,z);

global g L m c k;

Fi=z(1);
alpha=z(2);
alphadot=+g/L*sin(Fi)-c*alpha-k*Fi+0.5*sin(t)*cos(Fi);

zdot=[alpha alphadot]';
```





f) Suppose $F(t) = 0.5 ml \sin t$. the plot of ϕ vs. time is shown as above. As you can see, at the beginning, disturbing force won't affect much. But ~~with~~ later, it becomes dominant.