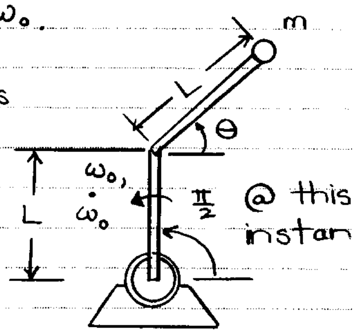
TAM 203 - Spring, 2000

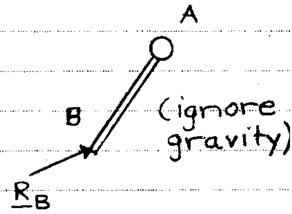
HW #14 Solutions : 5/13/00 by "Staff"

9.122 A motor @ O turns @ rate $\omega_o, \dot{\omega}_o$. At the end of a stick connected to this motor is a frictionless hinge attached to another massless stick. At the end of the 2^{ND} stick is a mass m. For the shown configuration, what is $\ddot{\theta}$?



AMB/B : $\Sigma \underline{M}_{/B} = \dot{\underline{H}}_{/B}$

FBD of 2^{ND} stick w/ mass



$$\Sigma \underline{M}_{/B} = \underline{O}$$

$$\dot{\underline{H}}_{/B} = \underline{r}_{A/B} \times m \underline{a}_A$$

Find $\underline{a}_A$:

$$\underline{a}_A = \underline{a}_B + \underline{a}_{A/B}$$

$$= \underline{\omega}_o \times (\underline{\omega}_o \times \underline{r}_{B/o}) + \dot{\underline{\omega}}_o \times \underline{r}_{B/o} + \dot{\theta}\hat{k} \times (\dot{\theta}\hat{k} \times \underline{r}_{A/B}) + \ddot{\theta}\hat{k} \times \underline{r}_{A/B}$$

$$= \omega_o \hat{k} \times [\omega_o \hat{k} \times L\hat{j}] + \dot{\omega}_o \hat{k} \times L\hat{j} + \dot{\theta}\hat{k} \times [\dot{\theta}\hat{k} \times L(\cos\theta \hat{i} + \sin\theta \hat{j})] + \ddot{\theta}\hat{k} \times L(\cos\theta\hat{i} + \sin\theta\hat{j})$$

$$= -\omega_o^2 L\hat{j} - \dot{\omega}_o L\hat{i} - \dot{\theta}^2 L(\cos\theta\hat{i} + \sin\theta\hat{j}) + \ddot{\theta}L(-\sin\theta\hat{i} + \cos\theta\hat{j})$$

$$= (-\dot{\omega}_o L - \dot{\theta}^2 L\cos\theta - \ddot{\theta}L\sin\theta)\hat{i} + (-\omega_o^2 L - \dot{\theta}^2 L\sin\theta + \ddot{\theta}L\cos\theta)\hat{j}$$

From AMB/B above, $\dot{\underline{H}}_{/B} = \underline{O}$

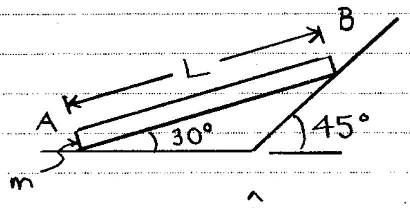
$$\Rightarrow \underline{r}_{A/B} \times m\underline{a}_A = \underline{O}$$

$$L(\cos\theta\hat{i} + \sin\theta\hat{j}) \times mL[-(\dot{\omega}_o + \dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta)\hat{i} + (-\omega_o^2 - \dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta)\hat{j}] =$$

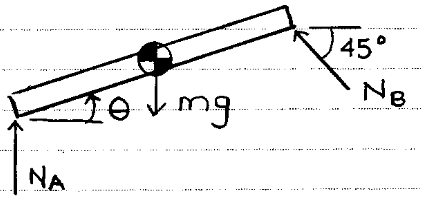
$$\Rightarrow \{mL^2[-\omega_o^2\cos\theta - \dot{\theta}^2\sin\theta\cos\theta + \ddot{\theta}\cos^2\theta + \dot{\omega}_o\sin\theta + \dot{\theta}^2\sin\theta\cos\theta + \ddot{\theta}\sin^2\theta]\hat{k} = \underline{O}$$

$$\{\}\cdot\hat{k} \Rightarrow \boxed{\ddot{\theta} = \omega_o^2\cos\theta - \dot{\omega}_o\sin\theta}$$

A uniform bar leans on a wall and floor and is let go from rest.



a) <u>FBD of the bar</u>



(Assume frictionless walls)

b) Find velocity & acceleration of B in terms of vel. & accel. of A:

$$\underline{v}_B = \underline{v}_A + \underline{v}_{B/A}$$

where
$$\underline{v}_B = v_B(\cos 45°\,\hat{\imath} + \sin 45°\,\hat{j})$$
$$\underline{v}_A = v_A\,\hat{\imath}$$
$$\underline{v}_{B/A} = \underline{\omega} \times \underline{r}_{B/A} = \omega\hat{k} \times L(\cos 30°\,\hat{\imath} + \sin 30°\,\hat{j}) = L\omega\left(-\tfrac{1}{2}\hat{\imath} + \tfrac{\sqrt{3}}{2}\hat{j}\right)$$

$$\Rightarrow \left\{ \tfrac{\sqrt{2}}{2} v_B(\hat{\imath}+\hat{j}) = v_A\hat{\imath} + \tfrac{1}{2}\omega L(-\hat{\imath} + \sqrt{3}\,\hat{j}) \right\}$$

$$\{\}\cdot\hat{j} \Rightarrow \tfrac{\sqrt{2}}{2} v_B = \tfrac{\sqrt{3}}{2}\omega L$$

$$\therefore \quad \omega = \sqrt{\tfrac{2}{3}}\,\frac{v_B}{L} \qquad (1)$$

$$\{\}\cdot\hat{\imath} \Rightarrow \tfrac{\sqrt{2}}{2} v_B = v_A - \tfrac{1}{2}\omega L \quad \text{(sub in (1))}$$
$$= v_A - \tfrac{1}{2}\left(\sqrt{\tfrac{2}{3}}\,\frac{v_B}{L}\right)L$$
$$= v_A - \tfrac{\sqrt{2}}{2} v_B\left(\tfrac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \tfrac{\sqrt{2}}{2} v_B = \frac{v_A}{1 + 1/\sqrt{3}}$$

$$\therefore \boxed{\underline{v}_B = \frac{v_A}{1 + 1/\sqrt{3}}(\hat{\imath} + \hat{j})} \qquad (2)$$

$$\underline{a}_B = \underline{a}_A + \underline{a}_{B/A}$$

where
$$\underline{a}_B = a_B(\cos 45°\,\hat{\imath} + \sin 45°\,\hat{j})$$
$$\underline{a}_A = a_A\,\hat{\imath}$$
$$\underline{a}_{B/A} = \underline{\omega} \times (\underline{\omega} \times \underline{r}_{B/A}) + \underline{\dot{\omega}} \times \underline{r}_{B/A}$$
$$= \omega\hat{k} \times [\omega\hat{k} \times L(\cos 30°\,\hat{\imath} + \sin 30°\,\hat{j})] + \dot{\omega}\hat{k} \times L(\cos 30°\,\hat{\imath} + \sin 30°\,\hat{j})$$
$$= -\tfrac{1}{2}L\omega^2(\sqrt{3}\,\hat{\imath} + \hat{j}) + \tfrac{1}{2}L\dot{\omega}(-\hat{\imath} + \sqrt{3}\,\hat{j})$$

(continued

$$\Rightarrow \left\{ \tfrac{\sqrt{2}}{2} a_B (\hat{i}+\hat{j}) = a_A \hat{i} - \tfrac{1}{2} L\omega^2 (\sqrt{3}\,\hat{i}+\hat{j}) + \tfrac{1}{2} L\dot{\omega}\,(-\hat{i}+\sqrt{3}\,\hat{j}) \right\}$$

$$\{\}\cdot\hat{j} \Rightarrow \tfrac{\sqrt{2}}{2} a_B = -\tfrac{1}{2}L\omega^2 + \tfrac{\sqrt{3}}{2} L\dot{\omega} \qquad (2)$$

$$\Rightarrow \dot{\omega} = \frac{1}{L\sqrt{3}}\left(\sqrt{2}\,a_B - L\omega^2\right)$$

$$\{\}\cdot\hat{i} \Rightarrow \tfrac{\sqrt{2}}{2} a_B = a_A - \tfrac{\sqrt{3}}{2} L\omega^2 - \tfrac{1}{2} L\dot{\omega}$$

$$\tfrac{\sqrt{2}}{2} a_B = a_A - \tfrac{\sqrt{3}}{2}L\omega^2 - \tfrac{1}{2}L\left(\frac{1}{L\sqrt{3}}\right)\left(\sqrt{2}\,a_B - L\omega^2\right)$$
$$= a_A - \tfrac{\sqrt{3}}{2}L\omega^2 - \tfrac{1}{\sqrt{3}}\left(\tfrac{\sqrt{2}}{2}a_B\right) + \tfrac{1}{2\sqrt{3}}L\omega^2$$

$$\tfrac{\sqrt{2}}{2} a_B \left(1 + \tfrac{1}{\sqrt{3}}\right) = a_A + L\omega^2\left(\tfrac{\sqrt{3}}{6} - \tfrac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \tfrac{\sqrt{2}}{2} a_B = \frac{a_A - \frac{1}{\sqrt{3}} L\omega^2}{1 + 1/\sqrt{3}} \qquad (3)$$

where $\omega = \sqrt{\tfrac{2}{3}}\,\frac{v_B}{L} = \sqrt{\tfrac{2}{3}}\left(\frac{v_A/\sqrt{2}}{1+1/\sqrt{3}}\right)\frac{1}{L}$ from (1)

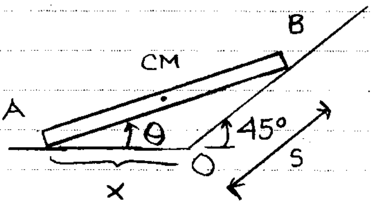
$$= \left(\frac{1}{1+\sqrt{3}}\right)\frac{v_A}{L}$$

$$\therefore \boxed{\underline{a}_B = \frac{\sqrt{3}\,a_A - L\omega^2}{1+\sqrt{3}}\,(\hat{i}+\hat{j})} \qquad (3)$$

c) Using the equations of motion find $\underline{a}_A$.

Use energy conservation (normal forces don't do any work)

$$E_K = \tfrac{1}{2} m v_{CM}^2 + \tfrac{1}{2} I_{zz}^{CM}\,\omega^2$$



$$x_{CM/O} = s\cos 45° - \tfrac{L}{2}\cos\theta$$
$$y_{CM/O} = \tfrac{L}{2}\sin\theta$$

where $\dfrac{\sin\theta}{s} = \dfrac{\sin 45°}{L}$

$$\Rightarrow s = \sqrt{2}\,L\sin\theta$$

$$\Rightarrow x_{CM/O} = L\sin\theta - \tfrac{1}{2}L\cos\theta$$
$$y_{CM/O} = \tfrac{1}{2}L\sin\theta$$

$$\Rightarrow \dot{x}_{CM/O} = L\dot{\theta}\cos\theta + \tfrac{1}{2}L\dot{\theta}\sin\theta$$
$$\dot{y}_{CM/O} = \tfrac{1}{2}L\dot{\theta}\cos\theta$$

So, $v_{CM}^2 = \dot{x}_{CM/O}^2 + \dot{y}_{CM/O}^2$

$$= L^2\dot{\theta}^2\left(\cos^2\theta + \sin\theta\cos\theta + \tfrac{1}{4}\sin^2\theta + \tfrac{1}{4}\cos^2\theta\right)$$

(continued)

$$\Rightarrow E_K = \tfrac{1}{2} m L^2 \dot{\theta}^2 \left(\tfrac{1}{4} + \cos^2\theta + \sin\theta\cos\theta \right)$$

$$E_P = mg\left(\tfrac{L}{2}\sin\theta\right) \qquad E_{TOT} = E_K + E_P \quad (4)$$

Energy conserved $\Rightarrow \dfrac{dE_{TOT}}{dt} = 0$

$$\Rightarrow \frac{d}{dt}\left[\tfrac{1}{2} m L^2 \dot{\theta}^2 \left(\tfrac{1}{4} + \cos^2\theta + \sin\theta\cos\theta \right) + \tfrac{1}{2} m g L \sin\theta \right] = 0$$

$$\Rightarrow mL^2 \dot{\theta}\ddot{\theta}\left(\tfrac{1}{4} + \cos^2\theta + \sin\theta\cos\theta \right) + \tfrac{1}{2} m L^2 \dot{\theta}^2 \left(-2\dot{\theta}\sin\theta\cos\theta + \dot{\theta}\cos^2\theta - \dot{\theta}\sin^2\theta \right) + \tfrac{1}{2} m g L \dot{\theta}\cos\theta = 0$$

$$L\ddot{\theta}\left(\tfrac{1}{4} + \cos^2\theta + \sin\theta\cos\theta \right) + \tfrac{1}{2} L \dot{\theta}^2 (\cos 2\theta - \sin 2\theta) + \tfrac{1}{2} g \cos\theta = 0$$

$$\Rightarrow \ddot{\theta} = \frac{\tfrac{1}{2} L \dot{\theta}^2 (\sin 2\theta - \cos 2\theta) - \tfrac{1}{2} g \cos\theta}{L\left(\tfrac{1}{4} + \cos^2\theta + \sin\theta\cos\theta \right)}$$

To get $\dot{\theta}$ @ $\theta = 30°$, go back to energy conservation:

$E_1 = E_2$ ← evaluate (4) @ $\theta = 30°$

$$\Rightarrow \tfrac{\sqrt{2}}{2} m g L = \tfrac{1}{2} m L^2 \dot{\theta}^2 \left(\tfrac{1}{4} + \cos^2 30° + \sin 30° \cos 30° \right) + \tfrac{1}{2} m g L \cos 30°$$

$$= \tfrac{1}{2} m L^2 \dot{\theta}^2 \left(\tfrac{1}{4} + \tfrac{3}{4} + \tfrac{\sqrt{3}}{4} \right) + \tfrac{\sqrt{3}}{4} m g L$$

$$= \tfrac{1}{2} m L^2 \dot{\theta}^2 \left(1 + \tfrac{\sqrt{3}}{4} \right) + \tfrac{\sqrt{3}}{4} m g L$$

$$g\left(\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{3}}{4} \right) = \tfrac{1}{2} L \dot{\theta}^2 \left(1 + \tfrac{\sqrt{3}}{4} \right)$$

$$\Rightarrow \dot{\theta}^2 = \frac{g\,(\sqrt{2} - \sqrt{3}/2)}{L(1 + \sqrt{3}/4)} = 0.3825\ g/L$$

$$\therefore \ddot{\theta} = \frac{\tfrac{1}{2} g \left(\frac{\sqrt{2} - \sqrt{3}/2}{1 + \sqrt{3}/4} \right)(\sin 60° - \cos 60°) - \tfrac{1}{2} g \cos 30°}{L\left(\tfrac{1}{4} + \cos^2 30° + \sin 30° \cos 30° \right)}$$

$$= \frac{g}{2L}\left[\frac{\left(\frac{\sqrt{2} - \sqrt{3}/2}{1 + \sqrt{3}/4} \right)\left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) - \frac{\sqrt{3}}{2}}{1 + \sqrt{3}/4} \right]$$

$$\ddot{\theta} = -0.2533\ g/L$$

Now find a_B from (2):

$$\tfrac{\sqrt{2}}{2} a_B = -\tfrac{1}{2} L \omega^2 + \tfrac{\sqrt{3}}{2} L \dot{\omega}$$

$$= -\tfrac{1}{2} L (0.3825\ g/L) + \tfrac{\sqrt{3}}{2} L (-0.2533\ g/L)$$

$$= -0.4106\ g$$

(continued)

$\Rightarrow a_A = \frac{\sqrt{3}}{2} a_B \left(1 + \frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}} L\omega^2$ from (3)

$= (-0.4106g)\left(1 + \frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}} L (0.3825\, g/L)$

$\Rightarrow a_A = -0.4268g$

$\therefore \underline{a}_A = -0.4268g\,\hat{i} = -4.187\ m/s^2\,\hat{i}$

d) Do the reaction forces add up to mg?

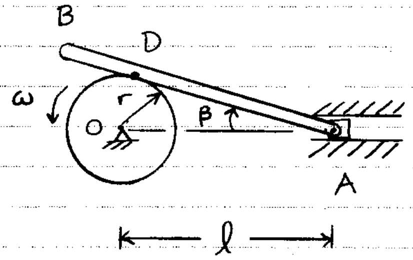
LMB: $\Sigma \underline{F} = m\underline{a}_{CM}$

$\underline{N}_A + \underline{N}_B - m\underline{g} = m\underline{a}_{CM}$

Since $\underline{a}_{CM} \neq \underline{0}$, the reaction forces do not add up to mg.

9.130 The link AB is supported by a wheel @ D and its end A is constrained so that it only has horiz. velocity.

Assume No slipping occurs between the wheel and the link.

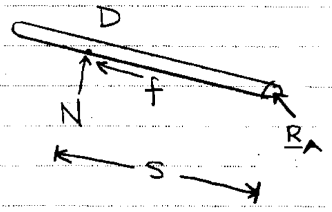


(continued)

Find the force on the wheel @ D due to the rod.

$\underline{AMB_{/A}}: \Sigma \underline{M}_{/A} = \dot{\underline{H}}_{/A}$

FBD of rod



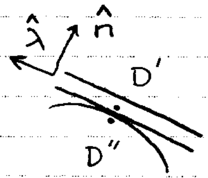
$\Sigma \underline{M}_{/A} = \underline{r}_{D/A} \times \underline{F}_A$
$= -Ns\,\hat{k}$

where $s^2 + r^2 = \ell^2$

$\Rightarrow \Sigma \underline{M}_{/A} = -N\sqrt{\ell^2 - r^2}\,\hat{k}$ (ignore gravity)

$\dot{\underline{H}}_{/A} = \underline{r}_{CM/A} \times m\underline{a}_{CM} + I\dot{\omega}\hat{k}$ $\quad (I \equiv I_{zz}^{CM})$

$\underline{\text{Need } \underline{a}_{CM}}$: Use the fact that there is no slip between the rod and the wheel and that the wheel rotates @ constant rate.



ω const. $\Rightarrow \underline{a}_{D''}$ has no tangential component

no slip $\Rightarrow \underline{a}_{D'} \cdot \hat{\lambda} = \underline{a}_{D''} \cdot \hat{\lambda}$
$= 0$

$\therefore \underline{a}_{D'} = a_{D'}\hat{n} = a_{D'}(\sin\beta\,\hat{\imath} + \cos\beta\,\hat{\jmath})$

Also know that $\underline{a}_A = a_A\hat{\imath}$

$\Rightarrow \underline{a}_{D'} = \underline{a}_A + \underline{a}_{D'/A}$

$\{a_{D'}(\sin\beta\,\hat{\imath} + \cos\beta\,\hat{\jmath}) = a_A\hat{\imath} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{D'/A}) + \dot{\underline{\omega}} \times \underline{r}_{D'/A}$

$= a_A\hat{\imath} + (-\dot{\beta}\hat{k}) \times [(-\dot{\beta}\hat{k}) \times s(-\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath})] + (-\ddot{\beta}\hat{k}) \times s(-\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath})$

$= a_A\hat{\imath} - s\dot{\beta}^2(-\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}) + s\ddot{\beta}(\sin\beta\,\hat{\imath} + \cos\beta\,\hat{\jmath})\}$

$\{\} \cdot \hat{\imath} \Rightarrow a_{D'}\sin\beta = a_A + s\dot{\beta}^2\cos\beta + s\ddot{\beta}\sin\beta$ (1)
$\{\} \cdot \hat{\jmath} \Rightarrow a_{D'}\cos\beta = -s\dot{\beta}^2\sin\beta + s\ddot{\beta}\cos\beta$ (2)

Multiplying (1) by $\cos\beta$ and subtracting from (2) multiplied by $\sin\beta$:

$0 = -s\dot{\beta}^2\sin^2\beta + s\ddot{\beta}\sin\beta\cos\beta$
$- a_A\cos\beta - s\dot{\beta}^2\cos^2\beta - s\ddot{\beta}\sin\beta\cos\beta$
$= -a_A\cos\beta - s\dot{\beta}^2$

(continued)

$$\Rightarrow a_A = \frac{-s\dot\beta^2}{\cos\beta} \quad \Rightarrow \underline{a}_A = \frac{-s\dot\beta^2}{\cos\beta}\hat{\imath}$$

Now find $\underline{a}_{cM}$:

$$\underline{a}_{cM} = \underline{a}_A + \underline{a}_{cM/A}$$
$$= \frac{-s\dot\beta^2}{\cos\beta}\hat{\imath} + \underline{\omega}\times(\underline{\omega}\times\underline{r}_{cM/A}) + \dot{\underline{\omega}}\times\underline{r}_{cM/A}$$
$$= \frac{-s\dot\beta^2}{\cos\beta}\hat{\imath} + s\dot\beta^2(\cos\beta\hat{\imath} - \sin\beta\hat{\jmath}) + s\ddot\beta(\sin\beta\hat{\imath} + \cos\beta\hat{\jmath})$$

Back to $\dot{\underline{H}}_{/A}$:

$$\dot{\underline{H}}_{/A} = \underline{r}_{cM/A}\times m\underline{a}_{cM} - I\ddot\beta\hat{k}$$
$$= s(-\cos\beta\hat{\imath} + \sin\beta\hat{\jmath})\times m\Big[\frac{-s\dot\beta^2}{\cos\beta}\hat{\imath} + s\dot\beta^2(\cos\beta\hat{\imath} - \sin\beta\hat{\jmath}) + s\ddot\beta(\sin\beta\hat{\imath} + \cos\beta\hat{\jmath})\Big] - I\ddot\beta\hat{k}$$
$$= ms^2\dot\beta^2\tan\beta\,\hat{k} - ms^2\ddot\beta\,\hat{k} - I\ddot\beta\,\hat{k}$$

$$AMB_{/A}\cdot\hat{k} \Rightarrow -Ns = ms^2\dot\beta^2\tan\beta - ms^2\ddot\beta - I\ddot\beta$$
$$\Rightarrow N = -ms\dot\beta^2\tan\beta + ms\ddot\beta + \frac{I}{s}\ddot\beta$$

Now, need to find $\dot\beta$, $\ddot\beta$:

no slip $\Rightarrow \underline{v}_{D'} = \underline{v}_{D''} = \omega r\hat{\lambda} = \omega r(-\cos\beta\hat{\imath} + \sin\beta\hat{\jmath})$

$$\Rightarrow \underline{v}_{D'} = \underline{v}_A + \underline{v}_{D'/A}$$
$$\{\omega r(-\cos\beta\hat{\imath} + \sin\beta\hat{\jmath}) = v_A\hat{\imath} + \underline{\omega}\times\underline{r}_{D'/A}$$
$$= v_A\hat{\imath} + (-\dot\beta\hat{k})\times s(-\cos\beta\hat{\imath} + \sin\beta\hat{\jmath})$$
$$= v_A\hat{\imath} + s\dot\beta(\sin\beta\hat{\imath} + \cos\beta\hat{\jmath})\}$$

$$\{\}\cdot\hat{\jmath} \Rightarrow \omega r\sin\beta = s\dot\beta\cos\beta \quad (3)$$
$$\Rightarrow \dot\beta = \frac{\omega r}{s}\tan\beta$$

To get $\ddot\beta$, can differentiate the above expression (3):

$$\omega r\dot\beta\cos\beta = s\ddot\beta\cos\beta - s\dot\beta^2\sin\beta + \dot s\dot\beta\cos\beta$$
$$\Rightarrow \ddot\beta = \frac{\omega r\dot\beta\cos\beta + s\dot\beta^2\sin\beta + \dot s\dot\beta\cos\beta}{s\cos\beta}$$

(continued)

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Can get $\dot{s}$ using $\tan\beta = \frac{r}{s}$:

$$s\tan\beta = r \Rightarrow \dot{s}\tan\beta + s\dot{\beta}\sec^2\beta = 0$$

$$\Rightarrow \quad \dot{s} = \frac{-s\dot{\beta}\sec^2\beta}{\tan\beta} = \frac{-s\dot{\beta}}{\sin\beta\cos\beta}$$

Putting it all together :

$$N = -ms\dot{\beta}^2\tan\beta + ms\ddot{\beta} + \frac{I}{s}\ddot{\beta}$$

where $s = \sqrt{l^2 - r^2}$

$$\dot{\beta} = \frac{\omega r}{s}\tan\beta ,$$

$$\ddot{\beta} = \frac{\omega r}{s}\dot{\beta} + \dot{\beta}^2\tan\beta + \frac{\dot{s}\dot{\beta}}{s}$$

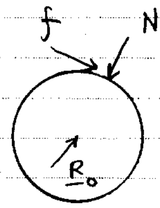
$$\dot{s} = \frac{-s\dot{\beta}}{\sin\beta\cos\beta}$$

$$\sin\beta = \frac{r}{l}, \quad \cos\beta = \frac{s}{l}$$

To find f, look @ FBD of the disk

AMB/o : $\Sigma \underline{M}_{/o} = \dot{\underline{H}}_{/o}$

where $\dot{\underline{H}}_{/o} = \underline{0}$ since ω is const.

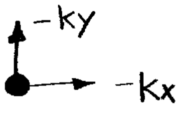


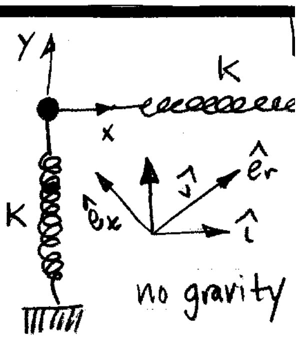
$$\Sigma \underline{M}_{/o} = -fr\hat{k} = \underline{0}$$

$$\therefore \boxed{f = 0}$$

9.140 FBD

by: J. Weisenfeld





A. LMB $\sum \underline{F} = m\underline{a}$

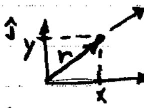
cartesian acceleration $\underline{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{J}$

$$-ky\hat{J} - kx\hat{\imath} = (\ddot{x}\hat{\imath} + \ddot{y}\hat{J})m$$

$$\boxed{-k(x\hat{\imath} + y\hat{J}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{J})}$$

can get cartesian acceleration just b[y] differentiating position vector

B. use LMB in polar coordinates, too, use polar coordinate conversions of answer in **A.**



in particular note that $r\hat{e}_r = x\hat{\imath} + y\hat{J}$

we know, too that most general acceleration in 2D-polar coordinates $\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$$\boxed{-kr\,\hat{e}_r = m(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_t}$$

C. use definition of angular momentum

$\underline{H} = \sum(\underline{r} \times \underline{V})m \qquad \underline{r} = x\hat{\imath} + y\hat{J} \qquad \underline{V} = \dot{x}\hat{\imath} + \dot{y}\hat{J}$

$\sum \underline{M}_{/o} = \dot{\underline{H}}_{/o}$ but $\sum \underline{M}_{/o} = r\hat{e}_r \times \underline{F} = r\hat{e}_r \times (-kr\hat{e}_r) = \underline{0}$

$\dot{\underline{H}}_{/o} = \frac{d}{dt}\left(m(x\dot{y} - y\dot{x})\right)\hat{k} = \underline{0} \Leftarrow \sum M_{/o} = 0 \Rightarrow \boxed{\frac{d}{dt}(x\dot{y} - y\dot{x}) = 0}$

D. use definition of angular momentum again

$\underline{H} = \sum(\underline{r} \times \underline{V})m \qquad \underline{r} = r\hat{e}_r \qquad \underline{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_t$

$\frac{d}{dt}\left[m\left(r\hat{e}_r \times (\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_t)\right)\right] = 0 \Rightarrow \boxed{\frac{d}{dt}(r^2\dot{\theta}) = 0}$

E. first, carry out the differentiation in **C.**

$\frac{d}{dt}(x\dot{y} - y\dot{x}) = \dot{x}\dot{y} + x\ddot{y} - \dot{y}\dot{x} - y\ddot{x} = x\ddot{y} - y\ddot{x} =$

notice that equation in **A.** dotted with $y\hat{\imath} - x\hat{J}$ or $(x\hat{J} - y\hat{\imath}$

gives you this $x\ddot{y} - y\ddot{x} = 0$

which you can integrate by parts to get **C.**

do the same for **D.** $\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} =$

notice that this equation is equation in **B.** dotted with $r\hat{e}_t$.

F. $E_K + E_P = \text{const.}$ is conservation of energy

$$\boxed{\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}k(x^2 + y^2) = \text{const}}$$

where we have used energy of spring $= \frac{1}{2}k(\Delta x)^2$

and the kinetic energy $= \frac{1}{2}m(\underline{V} \cdot \underline{V}) = \frac{1}{2}m(V)^2$

9.140 CONTINUED

[G.] $E_K + E_P$ = const is conservation of energy

$$E_K = \frac{1}{2}m(\underline{V}\cdot\underline{V}) = \frac{1}{2}m(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_t)\cdot(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_t)$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$
$$E_P = \frac{1}{2}k(\Delta x^2 + \Delta y^2) = \frac{1}{2}k(r^2\cos^2\theta + r^2\sin^2\theta)$$
$$= \frac{1}{2}kr^2$$

$$\boxed{\text{const} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}kr^2}$$

[H.] take the derivative of [F.] since energy is a constant you get $m\dot{x}\ddot{x} + m\dot{y}\ddot{y} + kx\dot{x} + ky\dot{y}$ so comparing to [A.] you must only dot [A.] with $\underline{V} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$ to get [F.]

now take the derivative of [G.] since energy is a consta[nt] you get $m\dot{r}\ddot{r} + mr\dot{\theta}(\dot{r}\dot{\theta} + r\ddot{\theta}) + kr\dot{r} = 0$ so comparing to [B.] you must only dot [B.] with $\underline{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$ to get [G.]

[I.] dotting original LMB in [A] by $\hat{\imath}$ and $\hat{\jmath}$ we get

$$\left.\begin{array}{l} m\ddot{x} + kx = 0 \\ m\ddot{y} + ky = 0 \end{array}\right\} \text{standard SHO} \left\{ \text{have solutions} \begin{array}{l} x = D\cos\omega t + E\sin\omega t \\ y = F\cos\omega t + G\sin\omega t \end{array}\right.$$

where $\omega = \sqrt{\frac{k}{m}}$ note: you can always write $x = D\cos\omega t + E\sin\omega t$,

$F = \sqrt{D^2+E^2}$, $G = \tan^{-1}\left(\frac{D}{E}\right)$ $x = F\sin(\omega t + G)$ or

$B = \sqrt{D^2+E^2}$, $C = \tan^{-1}\left(-\frac{E}{D}\right)$ $x = B\cos(\omega t + C)$

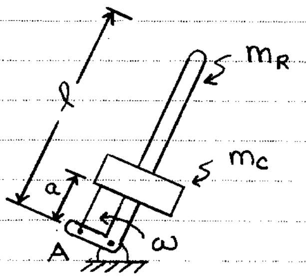
you can show that plots of y vs. x are ellipses, lines & circles (which may be rotated in the plane but let's imagine that x is the forcing function and y is the response on a damped simple harmonic oscillator (which mathematically they very well could be) then from Lab 1 we have seen that plots of y vs x are ellipses, circles and lines.

$x = A_1\sin(\omega t + B_1)$ forcing

phase lag = $B_2 - B_1$ $y = A_2\sin(\omega t + B_2)$ response

[J.] Using the analogy described above, we know that when a damped simple harmonic oscillator is driven at a small enough frequency, the response is in phase and has the same amplitude. So we should get the 45° line when $A_1 = A_2$ and $B_1 = B_2$. You will get a line of slope $\frac{A_1}{A_2}$ if $B_1 = B_2$, in fact.

A rod w/ mass m_R pivots w/o friction about pt. A. A collar w/ mass m_c slides w/o friction on the rod after the string connecting it to pt. A is cut. Before the string is cut the rod has angular velocity ω_1.



a) What is the speed of the collar as it flies off the end of the rod?

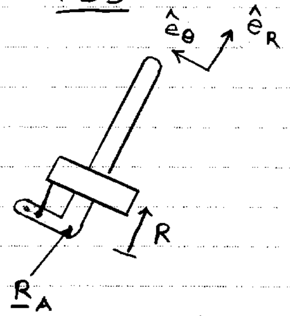
Consider the system consisting of the collar AND the rod:

With or without the string

$$\Sigma \underline{M}_{/A} = \underline{0}$$

(tension in the string is an internal force)

FBD



∴ from AMB/A :

$$\Sigma \underline{M}_{/A} = \dot{\underline{H}}_{/A} = \underline{0} \Rightarrow \underline{H}_{/A} \text{ conserved}$$

Before string is cut :

$$\underline{H}^{①}_{/A} = \underbrace{\underline{r}_{c/A} \times m_c \underline{v}_c}_{\underline{H}_{/A} \text{ due to collar}} + \underbrace{I^A_{zz}\, \omega_1 \hat{k}}_{\underline{H}_{/A} \text{ due to rod}}$$

$$= a\hat{e}_R \times m_c (a\omega_1 \hat{e}_\theta) + \tfrac{1}{3} m_R \ell^2 \omega_1 \hat{k}$$
$$= (m_c a^2 + \tfrac{1}{3} m_R \ell^2)\, \omega_1 \hat{k}$$

When collar is @ $R = \ell$:

$$\underline{H}^{②}_{/A} = \underline{r}_{c/A} \times m_c \underline{v}_c + I^A_{zz}\, \omega_2 \hat{k}$$
$$= \ell\hat{e}_R \times m_c (\dot{R}\hat{e}_R + \ell\omega_2 \hat{e}_\theta) + \tfrac{1}{3} m_R \ell^2 \omega_2 \hat{k}$$
$$= (m_c + \tfrac{1}{3} m_R)\ell^2 \omega_2 \hat{k}$$

$$\{\underline{H}^{①}_{/A} = \underline{H}^{②}_{/A}\} \cdot \hat{k}$$

$$\Rightarrow \boxed{\omega_2 = \frac{m_c a^2 + \frac{1}{3} m_R \ell^2}{(m_c + \frac{1}{3} m_R)\ell^2}\, \omega_1} \quad (1)$$

(continued)

Also note that energy is conserved
(no friction ⇒ no dissipation,
$\underline{R}_A$ does no work)

Before string is cut:

$$E_1 = \underbrace{\tfrac{1}{2} m_c V_c^2}_{\text{due to collar}} + \underbrace{\tfrac{1}{2} I_{zz}^A \omega_1^2}_{\text{due to rod}}$$

$$= \tfrac{1}{2} m_c (a\omega_1)^2 + \tfrac{1}{2}\left(\tfrac{1}{3} m_R \ell^2\right)\omega_1^2$$

$$= \tfrac{1}{2}\left(m_c a^2 + \tfrac{1}{3} m_R \ell^2\right)\omega_1^2$$

When collar is @ $R = \ell$:

$$E_2 = \tfrac{1}{2} m_c V_c^2 + \tfrac{1}{2} I_{zz}^A \omega_2^2$$

$$= \tfrac{1}{2} m_c V_c^2 + \tfrac{1}{2}\left(\tfrac{1}{3} m_R \ell^2\right)\omega_2^2$$

$$E_1 = E_2 \Rightarrow \tfrac{1}{2}\left(m_c a^2 + \tfrac{1}{3} m_R \ell^2\right)\omega_1^2 = \tfrac{1}{2} m_c V_c^2 + \tfrac{1}{6} m_R \ell^2 \omega_2^2$$

where ω_2 is given on the previous page

$$\omega_2 = \frac{m_c a^2 + \frac{1}{3} m_R \ell^2}{\left(m_c + \frac{1}{3} m_R\right)\ell^2}\,\omega_1$$

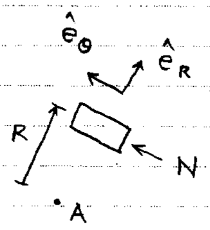
Plugging in: $m_R = 1$ kg, $m_c = 3$ kg, $a = 1$ m, $\ell = 3$ m, $\omega_1 = 1$ rad/s

$\omega_2 = 0.5$ rad/s , $\boxed{V_c = \frac{\sqrt{7}}{2}\ \frac{m}{s} = 1.323\ m/s}$

b) For the special case $m_R = 0$, sketch the path of the motion of the collar from the time the string is cut until some time after it leaves the end of the rod.

This is like the problem done in lecture.

FBD



LMB: $\Sigma \underline{F} = m_c \underline{a}_c$

$$N\hat{e}_\theta = m_c\left[(\ddot{R} - R\omega^2)\hat{e}_R + (R\dot{\omega} + 2\dot{R}\omega)\hat{e}_\theta\right]$$

$$LMB \cdot \hat{e}_R \Rightarrow \ddot{R} - R\omega^2 = 0$$

However, unlike in lecture, ω is not constant.

(continued)

le:
nd

From momentum conservation (1) w/ $l=R$

$$\omega_2 = \omega = \frac{a^2}{R^2}\omega_1 \quad \text{when} \quad m_R = 0$$

$$\therefore \omega^2 = \frac{a^4\omega_1^2}{R^4}$$

$$\Rightarrow \ddot{R} - \frac{a^4\omega_1^2}{R^3} = \ddot{R} - \frac{c^2}{R^3} = 0 \quad \text{where } c = a^2\omega_1$$

Let $x = R$, $y = \dot{R}$

$$\Rightarrow \left.\begin{array}{l} \dot{x} = y \\ \dot{y} = c^2/x^3 \end{array}\right\} \frac{dy}{dx} = \frac{c^2/x^3}{y}$$

$$\Rightarrow \int y\,dy = \int c^2 \frac{dx}{x^3}$$

$$\frac{1}{2}y^2 = -\frac{c^2}{2}\frac{1}{x^2} + K_1$$

$$\frac{1}{2}\dot{R}^2 = -\frac{1}{2}c^2/R^2 + K_1$$

When $R = a$, $\dot{R} = 0 \Rightarrow K_1 = \frac{1}{2}\frac{c^2}{a^2}$

$$\Rightarrow \frac{1}{2}\dot{R}^2 = -\frac{1}{2}\frac{c^2}{R^2} + \frac{1}{2}\frac{c^2}{a^2}$$

$$\dot{R}^2 = c^2\left(\frac{1}{a^2} - \frac{1}{R^2}\right)$$

$$\Rightarrow \frac{dR}{dt} = c\sqrt{\frac{1}{a^2} - \frac{1}{R^2}} = \frac{c}{aR}\sqrt{R^2 - a^2}$$

$$\int \frac{R\,dR}{\sqrt{R^2 - a^2}} = \int \frac{c}{a}\,dt$$

$$\sqrt{R^2 - a^2} = \frac{c}{a}t + K_2$$

When $t = 0$, $R = a \Rightarrow K_2 = 0$

$$\Rightarrow R^2 = a^2 + \frac{c^2}{a^2}t^2$$

$$= a^2 + \frac{a^4\omega_1^2}{a^2}t^2$$

$$= a^2(1 + \omega_1^2 t^2)$$

$$\therefore \boxed{R(t) = a\sqrt{1 + \omega_1^2 t^2}}$$

Get $\theta(t)$:

$$\frac{d\theta}{dt} = \omega = \frac{a^2}{R^2}\omega_1 = \frac{\omega_1}{1 + \omega_1^2 t^2}$$

$$\Rightarrow \boxed{\theta(t) = \mathrm{Tan}^{-1}(\omega_1 t)} \quad (\text{let } \theta_0 = 0)$$

$$\Rightarrow \omega_1 t = \tan\theta \Rightarrow \boxed{R(\theta) = a\sec(\theta)}$$

(continued

Note that $x = R\cos\theta$, $y = R\sin\theta$

$$\Rightarrow \begin{array}{l} x = (a\sec\theta)\cos\theta = a \quad \leftarrow \text{Constant!} \\ y = (a\sec\theta)\sin\theta = a\tan\theta \end{array}$$

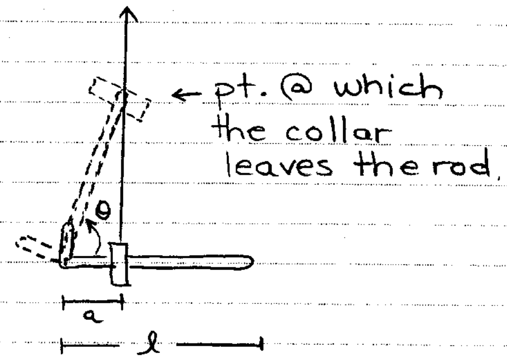
Sketch of the collar's trajectory:

$a = 1\,m$
$l = 3\,m$

leaves when

$\cos\theta = \frac{a}{l} = \frac{1}{3}$

OR $\theta = 70.5°$



When the collar leaves the rod, its trajectory will still be a straight line since no forces act on it anymore (no gravity).