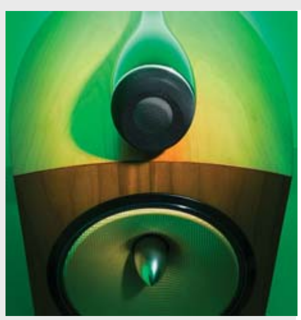
M&AE 2030: Dynamics Laboratory

Characteristics and Characterizing 1 d-o-f Vibrating Systems

- Example Systems
- Modeling; Measurements using Displacement Sensors
- Governing Equation and Solutions
- Transient Solution: Features and Application to Inverse Problem
- Steady-state Solution: Frequency Characteristics
- Resonance Curve: Analysis and Application to Inverse Problem

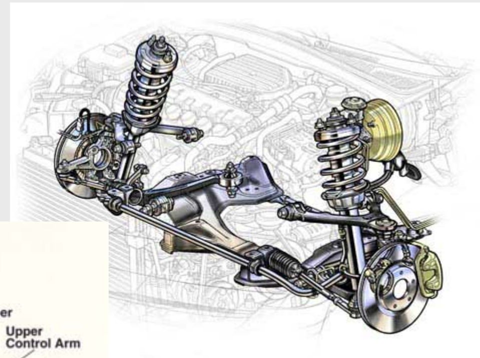
Approximate One d-o-f Vibrating Systems:

Loudspeakers,
Headphones,
Microphones…

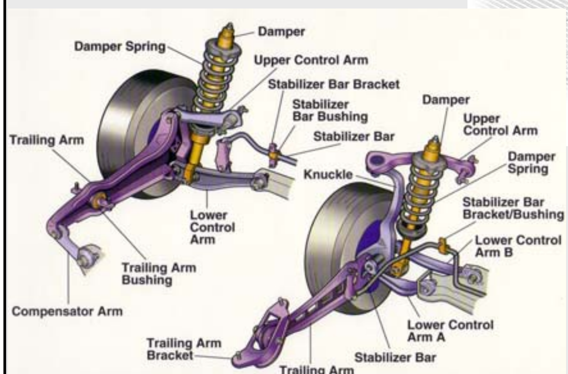



Approximate One d-o-f Vibrating Systems:

Front Suspension:



Rear Suspension:



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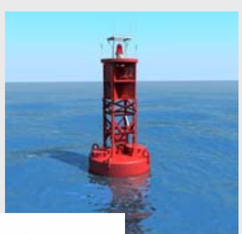
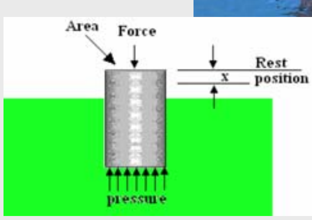
Approximate One d-o-f Vibrating Systems:

Floating Buoys:





Pendulum-based Clocks

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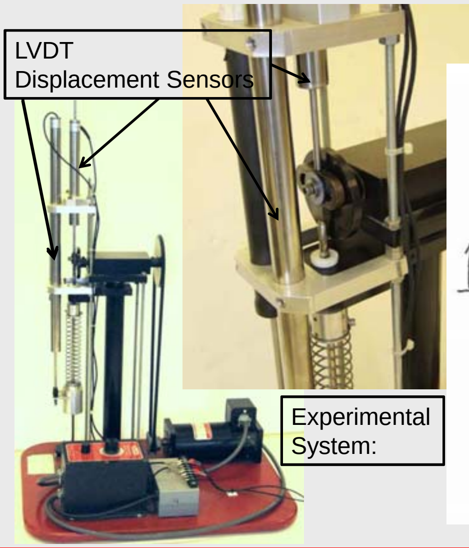
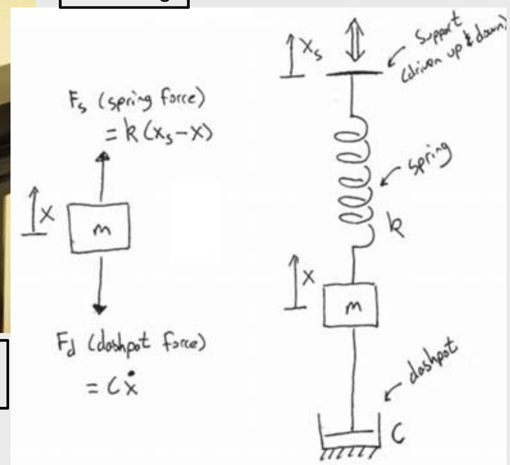
M&AE 2030: One Degree-of-Freedom Oscillator:

System and Modeling:



Modeling:



Displacement Sensing

- LVDT (Linear Variable Differential Transformer):
  - Inductance-based ctromechanical sensor
  - "Infinite" resolution
    - limited by external electronics
  - Limited frequency bandwidth (250 Hz typical for DC-LVDT, 500 Hz for AC-LVDT)
  - No contact between the moving core and coil structure
    - no friction, no wear, very long operating lifetime
  - Accuracy limited mostly by linearity
    - 0.1%-1% typical
  - Models with strokes from mm's to 1 m available

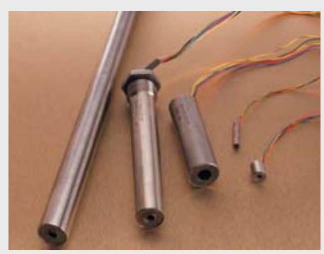
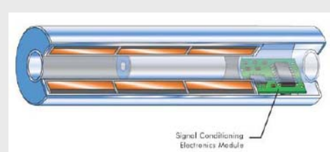
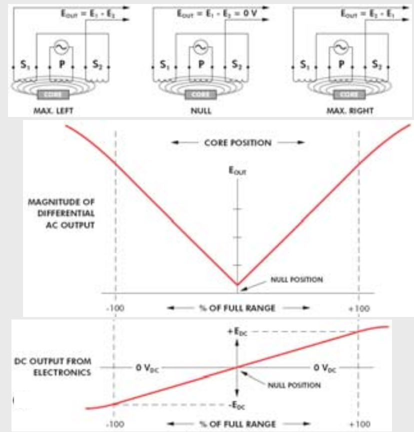


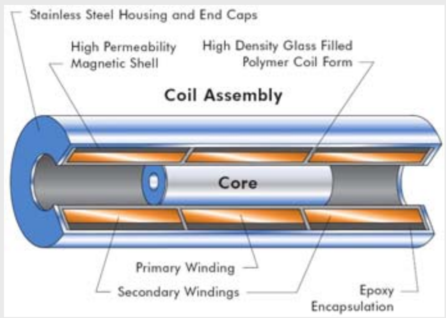
Photo courtesy of MSI



Linear Variable Differential Transformer - LVDT

- Attached to Object being sensed
- Measures linear position (displacement)
- Dynamic Range: 10^{-6} in → +/-20 in
- Infinite Resolution
- Signal Out: Several Volts-rms





Governing Equation: 1 d-o-f Vibrating System

Governing equations. The two forces we consider are:

$$F_{sp}(t) = k\,(x_s - x) \qquad = \text{The spring force} \qquad (1.1)$$
$$F_d(t) = c\dot{x} \qquad = \text{The dashpot force} \qquad (1.2)$$

From Newton's second law the equation of motion for this system is

$$\left\{\sum \underline{\mathbf{F}}\right\} \cdot \hat{\mathbf{e}}_{\mathbf{x}} \Rightarrow -F_d + F_{sp} = m\ddot{x} \qquad (1.3)$$

and plugging in for spring and dashpot terms we get

$$m\ddot{x} = -c\dot{x} + kx_s - kx. \qquad (1.4)$$

Rearranging we get the standard form

$$m\ddot{x} + c\dot{x} + kx = F_s(t) \quad \text{with} \quad F_s(t) = kx_s(t) \qquad (1.5)$$

where $F_s(t)$ is the (presumably specified) "forcing function" due to the motion of the support. In this case the forcing is from the end of the spring being displaced.

Solution: 1 d-o-f Vibrating System:

Equation of Motion: $m\ddot{x} + c\dot{x} + kx = F_s(t)$

Free-vibration case: $F_s(t) = 0 \rightarrow x_c(t)$ Transient Solution

Forced Vibration case: $F_s(t) = kx_s(t) = kA_{support}\cos\omega t$

$\rightarrow x_p(t)$ Particular Solution

General Solution: $x(t) = x_c(t) + x_p(t)$

$c^2 - 4mk > 0$ Overdamped motion; Non-periodic

$c^2 - 4mk = 0$ Critically-damped motion; Non-periodic

$c^2 - 4mk < 0$ Under-damped motion; Periodic; Decay

. . . All approach → $x_c(t) = 0$

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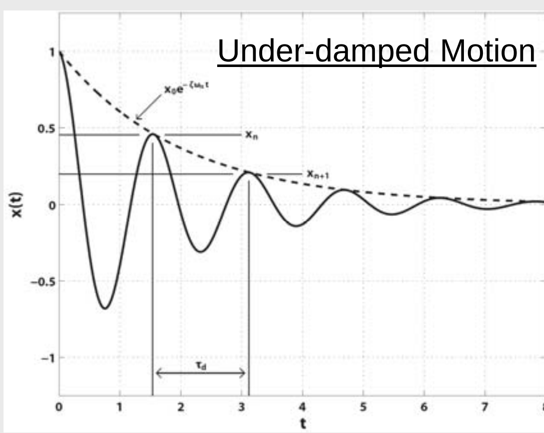
Transient Solution Features: 1 d-o-f System



Natural Frequency:

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

Damping Ratio:

$$\boxed{\zeta = \frac{c}{2\sqrt{mk}}}$$

Damped Frequency:

$$\boxed{\omega_d = \omega_n\sqrt{1 - \zeta^2}}$$

Solution:

$$x_c(t) = e^{-\zeta\omega_n t}\left[C_1\cos(\omega_d t) + C_2\sin(\omega_d t)\right]$$

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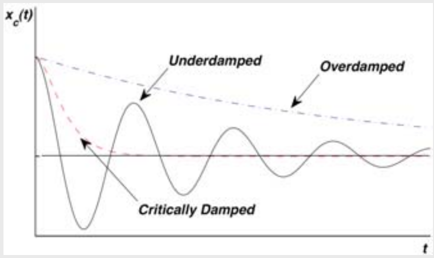
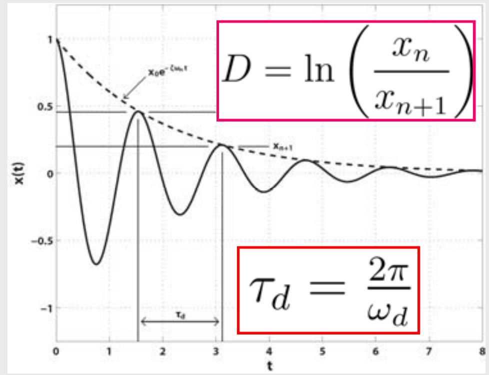
Transient Solution Features: 1 d-o-f System





$$\boxed{D = \ln\left(\frac{x_n}{x_{n+1}}\right)}$$

$$\boxed{\tau_d = \frac{2\pi}{\omega_d}}$$

Damped Frequency:

$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

Natural Frequency:

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

Damping Ratio:

$$\boxed{\zeta = \frac{c}{2\sqrt{mk}}}$$

$$D = \ln\left(\frac{e^{-\zeta\omega_n t}}{e^{-\zeta\omega_n(t+\tau_d)}}\right)$$

$$\boxed{= \zeta\omega_n\tau_d}$$

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Solution to the Inverse Problem:

Finding the moving mass m: Using the definitions of ζ and D and solving for m:

$$c = 2\sqrt{mk}\,\zeta = 2\sqrt{mk}\,\frac{D}{\tau_d\sqrt{\frac{k}{m}}} = \frac{2m\,D}{\tau_d} \Longrightarrow m = \frac{c\,\tau_d}{2\,D}$$

Carry out two *ring down* measurements. One with the original system and measure the quantities τ_d and D. Add a small, known mass, Δm to the moving mass and repeat the measurements. This time the measured quantities will be τ_d' and D'. Then form the ratio and solve for the moving mass m.

$$\boxed{m = \frac{\tau_d\,D'}{\tau_d'\,D - \tau_d\,D'}\,\Delta m}$$

Finding the damping c: Using the definitions of ζ and D and solving for c:

$$c = 2\sqrt{mk}\,\zeta = 2\sqrt{mk}\,\frac{D}{\tau_d\sqrt{\frac{k}{m}}} = \boxed{\frac{2m\,D}{\tau_d} = c}$$

Make a *ring down* measurement and extract the the logarithmic decrement D and τ_d, the period of the damped motion.

Finding the system stiffness k: Use the previously obtained values of damping c and moving mass m, then use the measured logarithmic decrement D and

$$\boxed{k = \frac{c^2\left(1+\frac{4\pi^2}{D^2}\right)}{4\,m}}$$

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Steady-state Solution:1d-o-f Vibrating System:

Equation of Motion: $m\ddot{x} + c\dot{x} + kx = F_s(t)$

Forced Vibration case: $F_s(t) = kx_s(t) = kA_{support} \cos \omega t$

→ $x_p(t)$ Particular (steady-state) Solution

will be of form … $x_p(t) = A_{response} \cos(\omega t - \phi)$

$A_{response}$: Output Amplitude of response

ϕ : Phase response

ω : Excitation frequency

Steady-state Solution: Frequency Characteristics

- The non-dimensional result for the response amplitude and its phase relative to the excitation is

$$\frac{A_{response}}{F_s/k} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

- Th… ou…

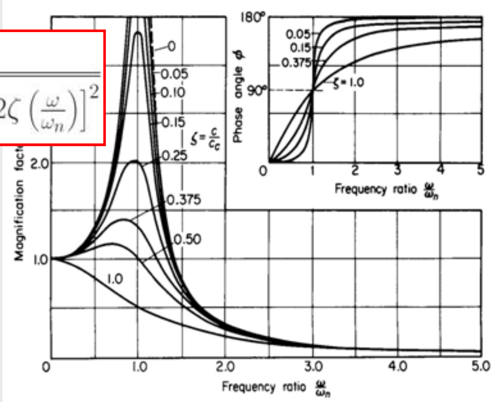
$$\tan\phi = \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_m}\right)^2}$$

Observ…

$\omega/\omega_n \ll 1$: Both the inertia and damping forces acting on the mass are small which results in a small phase angle and the magnitude of the excitation is nearly equal to the spring force.

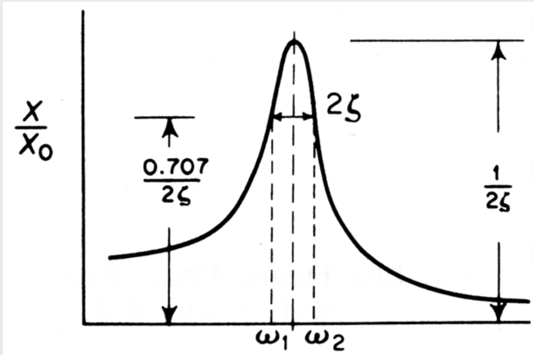
$\omega/\omega_n \approx 1$: The phase angle is 90°. Here the inertia force is balanced by the spring force while the applied force overcomes the damping force.

$\omega/\omega_n \gg 1$: The phase angle approaches 180° and the applied force is used almost entirely to overcome the large inertia force of the moving mass.



Frequency Response Curves: Magnitude and Phase

Analysis of the Resonance Curve:



- The maximum amplitude of the peak occurs approximately at frequency $\omega \approx \omega_n = \sqrt{k/m}$ (where $\omega = 2\pi f$; f is the frequency [Hz].)
- The maximum amplitude of the peak $A^{max}_{response}$ has value $(F_s/k)/2\zeta$.
- The frequency interval $(\omega_2 - \omega_1)$ at the resonance curve amplitudes $A^{max}_{response}/\sqrt{2}$ is given by $2\zeta\omega_n$.
- The *Quality Factor* Q of the system is then given by $Q = \omega_n/(\omega_2 - \omega_1)$ which is equivalent to $Q = 1/2\zeta$. This is another way of determining the damping of the system.

Solving the Inverse Problem:

1. Find resonance frequency, ω_n
2. Use a Δm, find ω_n'. Determine m.
3. Use m and ω_n to determine k.
4. Measure Q to find c ($=m\omega_n/Q$).

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